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Survey of Planar and Outerplanar Graphs in Fuzzy and Neutrosophic Graphs

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Abstract: As many readers may know, graph theory is a fundamental branch of mathematics that explores networks made up of nodes and edges, focusing on their paths, structures, and properties [196]. A planar graph is one that can be drawn on a plane without any edges intersecting, ensuring planarity. Outerplanar graphs, a subset of planar graphs, have all their vertices located on the boundary of the outer face in their planar embedding. In recent years, outerplanar graphs have been formally defined within the context of fuzzy graphs. To capture uncertain parameters and concepts, various graphs such as fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been studied. In this paper, we investigate planar graphs, outerplanar graphs, apex graphs, and others within the frameworks of neutrosophic graphs, Turiyam Neutrosophic graphs, fuzzy graphs, and plithogenic graphs.

Keywords: Neutrosophic graph, Turiyam Neutrosophic graph, Fuzzy graph, Planar graph, Outerplanar graph

1. Introduction

1.1 Graph Theory

As readers may be aware, graph theory, a fundamental branch of mathematics, explores networks composed of nodes and edges, focusing on their paths, structures, and properties [196]. Using graphs allows for a clear representation of connections between real-world concepts. This is one reason why graph theory has been extensively studied across a wide range of applications [61, 114, 182, 233, 271, 436, 528]. For instance, in the field of AI, research on graph neural networks, which employs graph theory concepts, is highly active [219, 475, 550, 552, 563].

1.2 Graph Class

One of the powerful tools in graph theory is the classification of graphs into distinct graph classes based on shared properties or structural features. Numerous graph classes have been introduced, each designed to capture specific behaviors or characteristics of a graph. These classes serve as a foundation for developing efficient algorithms, simplifying problem-solving, and gaining deeper insights into computational complexity (cf.[63, 116, 126, 415]).

Examples of well-known graph classes include Tree Graphs [503], Path Graphs [555], Complete Graphs [154], Trapezoid graph[168] Circle Graphs [121], Unit Disk Graphs [164], Circular-arc graph[251] Edge-Transitive Graphs [387, 388], Ultrahomogeneous Graphs [351], Visibility Graphs [350], Interval graphs [243], Caylay graphs [331], Comparability graph[549], Proper Interval graphs [148], Dense graphs[372], Universal graph[159], Outerplanar Graphs [296], Petersen Graphs [291], and Total Graphs [522]. The study of such graph classes allows researchers to identify common properties, develop more specialized and efficient algorithms, and apply these insights to practical problems. In line with these observations, a growing area of research focuses on analyzing how closely a given graph or algorithm matches desired graph classes or structures. (cf.[196,238,423,473,527]).

1.3 Planar graph and Outerplanar graph

In the field of graph theory, various graph classes with diverse properties are continuously studied and introduced. This paper primarily focuses on planar graphs and outerplanar graphs. A planar graph is defined as a graph that can be drawn on a plane without any edges crossing, thereby maintaining planarity [177,305,418,455, 478]. A notable example of a planar graph is the Butterfly graph (cf.[539]). An outerplanar graph, on the other hand, is a planar graph in which all vertices are positioned on the boundary of the outer face in its embedding [77, 296, 409]. Additionally, another well-known graph closely related to planar graphs is the Apex Graph. Due

to the edge-crossing restrictions of planar graphs, they are not only visually intuitive but are also often used in various papers for simplification purposes.

Research on parameter values of outerplanar graphs is ongoing. For instance, outerplanar graphs have a degeneracy of at most 2[365] [1] Additionally, outerplanar graphs have a boxicity [2] of at most 2 [476].

Other related graph classes include Halin graphs [113], upward planar graphs[16, 74, 316], convex planar graphs[187], 1-planar graph[292, 515], and k-Planar Graphs[84] which are also known in the context of planar graphs.

One of the key motivations for studying these specific graph classes is the development of efficient algorithms tailored to restricted graph types. For instance, similar to outerplanar graphs, Halin graphs exhibit low treewidth, which makes many algorithmic problems easier to solve compared to unrestricted planar graphs [516]. Numerous other studies have also examined planar graphs and their related graph classes, contributing significantly to this area of research[46, 132, 166, 342, 451].

1.4 Fuzzy Graphs and Neutrosophic Graphs

To address the uncertainties present in the world, various types of graphs like fuzzy, neutrosophic, Turiyam, and plithogenic graphs have been studied.

A fuzzy graph assigns a membership degree between 0 and 1 to each edge and vertex, representing the level of uncertainty. Simply put, a fuzzy graph is a graphical representation of a fuzzy set (cf. [206, 360, 558, 559]). In real-world applications, fuzzy graphs are used in areas such as social networks, decision-making, and transportation systems to model relationships or connections that are not precise or are uncertain [401, 459]. This wide range of applicability has led to significant research interest in fuzzy graphs.

Recently, neutrosophic graphs [28, 33, 130, 238, 276, 297, 319, 463, 493, 497] and neutrosophic hypergraphs [27, 37, 186, 373] have also gained attention within the framework of neutrosophic set theory [48, 501]. ”Neutrosophic” extends classical and fuzzy logic by introducing degrees of truth, indeterminacy, and falsity, offering a more comprehensive way to handle uncertainty.

Following this, the Turiyam Neutrosophic graph was introduced as a further extension of neutrosophic and fuzzy graphs. A Turiyam Neutrosophic graph stands out by assigning four attributes to each vertex and edge: truth, indeterminacy, falsity, and a liberal state, thus broadening the framework of neutrosophic and fuzzy graphs [244–246]. Additionally, plithogenic graphs have emerged as a more generalized form and are currently a subject of active study [321, 489, 506, 511].

Focusing on graph types that handle these uncertainties, particularly those related to planar graphs, we have examples such as fuzzy planar graphs [20, 442, 472], fuzzy outerplanar graphs [309], intuitionistic fuzzy planar graphs [49], and neutrosophic planar graphs [37, 146, 375].

1.5 Our Contribution

Building on the observations above, the study of planar graphs, outerplanar graphs, fuzzy graphs, and neutrosophic graphs is of great importance; however, these topics have not yet been fully explored. In this paper, we focus on neutrosophic outerplanar graphs and related graph classes. Our goal is to stimulate further research and development in the study of various graph classes.

The following diagram illustrates a portion of the overview of this paper. While this diagram focuses on planar graphs and apex graphs, a similar analysis will be conducted for outerplanar graphs and other related types.

2. Preliminaries and definitions

In this section, we present a brief overview of the definitions and notations used throughout this paper. We will specifically cover fundamental concepts related to graphs, including planar graphs, outerplanar graphs, apex graphs, fuzzy graphs, intuitionistic fuzzy graphs, Turiyam Neutrosophic graphs, neutrosophic graphs, and plithogenic graphs.

Additionally, please note that this paper may also incorporate concepts from set theory alongside graph theory. For a more comprehensive understanding of set theory, you may refer to the relevant surveys or notes [285, 311, 356].

2.1 Basic Graph Concepts

Here are a few basic graph concepts listed below. For more foundational graph concepts and notations, please refer to [142, 194, 195, 195, 196, 271, 283, 543, 547].

[1] Treewidth measures how closely a graph resembles a tree, representing the smallest ”width” of a tree decomposition [107–110, 115, 453, 454, 457].

[2] The boxicity of a graph is the minimum number of dimensions needed to represent it as an intersection graph of axis-aligned boxes [9, 139].

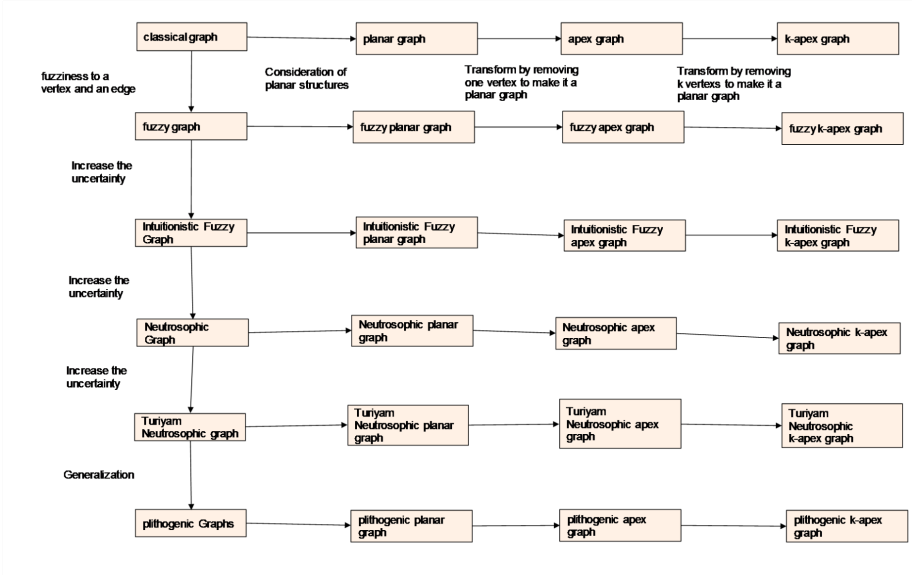

Fig. 1. Graph Hierarchy in this paper

Definition 1 (Graph)**.** [196] A graph G is a mathematical structure consisting of a set of vertices $V(G)$ and a set of edges $E(G)$ that connect pairs of vertices, representing relationships or connections between them. Formally, a graph is defined as $G = (V, E)$, where V is the vertex set and E is the edge set.

Definition 2 (Subgraph)**.** [196] A subgraph of G is a graph formed by selecting a subset of vertices and edges from G.

Example 3 (Example of a Subgraph)**.** Consider a graph $G = (V, E)$ with the vertex set $V = \{1, 2, 3, 4, 5\}$ and the edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}, \{4, 5\}\}.$$

This graph G can be visualized as having 5 vertices connected by 5 edges.

Now, let us define a subgraph G' by selecting the subset of vertices $V' = \{1, 2, 4\}$ and the subset of edges $E' = \{\{1, 2\}, \{2, 4\}\}$. Thus, the subgraph $G' = (V', E')$ has:

- The vertex set $V' = \{1, 2, 4\}$,
- The edge set $E' = \{\{1, 2\}, \{2, 4\}\}$.

In this example, G' is a valid subgraph of G because both $V' \subseteq V$ and $E' \subseteq E$, and the edges in E' only connect vertices within V'.

Definition 4 (Degree)**.** [196] Let $G = (V, E)$ be a graph. The *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v. Formally, for undirected graphs:

$$\deg(v) = |\{e \in E \mid v \in e\}|.$$

In the case of directed graphs, the *in-degree* $\deg^-(v)$ is the number of edges directed into v, and the *out-degree* $\deg^+(v)$ is the number of edges directed out of v.

Definition 5. (cf.[279, 531]) A graph $G = (V, E)$ is said to be a *connected graph* if for any two distinct vertices $u, v \in V$, there exists a path in G that connects u and v. In other words, every pair of vertices in the graph is reachable from each other, meaning there is a sequence of edges that allows traversal between any two vertices.

Mathematically, for all $u, v \in V$, there exists a sequence of vertices $v_1 = u, v_2, \ldots, v_k = v$ such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$.

Definition 6 (Path)**.** (cf.[560]) A path in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, 2, \ldots, k-1$. A path is represented as:

$$P = (v_1, v_2, \ldots, v_k),$$

where no vertex is repeated. The length of a path is the number of edges it contains, i.e., $k-1$.

Definition 7 (Tree)**.** (cf.[560]) A tree is a connected, acyclic graph. In other words, a tree is a graph where there is exactly one path between any two vertices, and no cycles exist.

Example 8. Consider a graph with 4 vertices $V = \{V_A, V_B, V_C, V_D\}$ and 3 edges $E = \{(V_A, V_B), (V_B, V_C), (V_C, V_D)\}$. This graph is connected, has no cycles, and there is exactly one path between any pair of vertices. Hence, this graph is a tree.

Definition 9. (cf.[367, 399, 440]) A Forest is an undirected graph that is acyclic, meaning it contains no cycles. It consists of one or more disjoint trees, where each tree is a connected subgraph with no cycles.

Formally, a graph $F = (V, E)$ is a forest if:

- V is the set of vertices.
- $E \subseteq V \times V$ is the set of edges.
- Each connected component of F is a tree, i.e., there is no path in any component that returns to the starting vertex, which implies no cycles exist in F.

In simpler terms, a forest is a collection of trees, and if a forest consists of only one connected component, it is itself a tree.

Example 10. Consider a graph with 6 vertices

$$V = \{V_A, V_B, V_C, V_D, V_E, V_F\}$$

and 4 edges

$$E = \{(V_A, V_B), (V_B, V_C), (V_D, V_E), (V_E, V_F)\}$$

. The graph consists of two disjointcomponents: one component with vertices $\{V_A, V_B, V_C\}$ and edges (V_A, V_B), (V_B, V_C), and another component with vertices $\{V_D, V_E, V_F\}$ and edges (V_D, V_E), (V_E, V_F). Each component is a tree, and thus, the entire graph is a forest.

Definition 11 (Complete Graph)**.** (cf.[83, 204]) A *complete graph* is a graph $G = (V, E)$ in which every pair of distinct vertices is connected by a unique edge. Formally, a graph $G = (V, E)$ is complete if for every pair of vertices $u, v \in V$ with $u \neq v$, there exists an edge $\{u, v\} \in E$.

The complete graph on n vertices is denoted by K_n, and it has the following properties:

- The number of vertices is $|V| = n$.
- The number of edges is $|E| = \binom{n}{2} = \frac{n(n-1)}{2}$.
- Each vertex has degree $\deg(v) = n-1$ for all $v \in V$.

Example 12 (Examples of Complete Graphs)**.** The concept of a complete graph is best understood through specific examples:

- The complete graph K_1 consists of a single vertex with no edges.
- The complete graph K_2 has two vertices, v_1 and v_2, and a single edge connecting them, $\{v_1, v_2\}$.
- The complete graph K_3 consists of three vertices, v_1, v_2, and v_3, with edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, and $\{v_1, v_3\}$. This forms a triangle.
- The complete graph K_4 has four vertices, with every possible pair of vertices connected by an edge. The edges are $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, and $\{v_3, v_4\}$. This graph forms a tetrahedron when represented in three dimensions.
- The complete graph K_5 includes five vertices, with each pair of vertices connected by a unique edge. The total number of edges is $\binom{5}{2} = 10$, making it a highly connected structure.

In general, for a complete graph K_n with n vertices, there are $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, and each vertex has a degree of $n-1$.

Definition 13 (Bipartite Graph)**.** (cf.[69,207]) A *bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- Every edge in E connects a vertex from V_1 to a vertex from V_2. In other words, there are no edges connecting two vertices within the same subset V_1 or V_2.

Formally, $G = (V, E)$ is bipartite if there exists a partition (V_1, V_2) such that for every edge $e = \{u, v\} \in E$, either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$.

A graph G is bipartite if and only if it contains no odd-length cycles.

Definition 14 (Complete Bipartite Graph)**.** (cf.[215,286,361]) A *complete bipartite graph* is a graph $G = (V, E)$ whose vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that:

- $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$.
- There is an edge between every vertex in V_1 and every vertex in V_2.
- There are no edges between vertices within the same subset V_1 or V_2.

The complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$. It has the following properties:

- The number of vertices is $|V| = m + n$.
- The number of edges is $|E| = m \times n$.
- Each vertex in V_1 has degree n, and each vertex in V_2 has degree m.

Example 15 (Examples of Bipartite Graphs)**.** To illustrate the concept of bipartite graphs, we provide the following examples:

- The simplest example is the graph $K_{1,1}$, which consists of two vertices v_1 and v_2 with a single edge connecting them. Here, $V_1 = \{v_1\}$ and $V_2 = \{v_2\}$, making $K_{1,1}$ a bipartite graph.
- The complete bipartite graph $K_{2,3}$ consists of two sets of vertices, $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1, v_2, v_3\}$. Each vertex in V_1 is connected to every vertex in V_2. Thus, the edge set E consists of all possible edges between V_1 and V_2, specifically: $\{u_1, v_1\}, \{u_1, v_2\}, \{u_1, v_3\}, \{u_2, v_1\}, \{u_2, v_2\}, \{u_2, v_3\}$. There are no edges within V_1 or V_2, making $K_{2,3}$ a bipartite graph.

Definition 16. (cf.[117,203,218,287,544]) Two graphs $G = (V, E)$ and $H = (V', E')$ are said to be *homomorphic* if there exists a mapping $\phi : V \to V'$ such that for every edge $(u, v) \in E$, the image $(\phi(u), \phi(v))$ is an edge in E'. In other words, there is a structure-preserving mapping from G to H that maintains the adjacency relationships between vertices.

2.2 Classic class about planar graph and outerplanar graph

We introduce some representative classical graph classes. These graph classes have also been studied in the contexts of fuzzy, neutrosophic, and plithogenic settings.

2.2.1 Planar graph

Planar graphs, co-planar graphs, and k-planar graphs have been extensively studied, with numerous applications in networks and other fields. The definitions are provided below.

Definition 17. (cf.[210,314]) An Edge Crossing occurs when two edges in a graph's drawing intersect at a point other than a common endpoint. In a planar representation of the graph, edge crossings indicate that the graph is drawn in such a way that edges overlap or cross each other, violating planarity.

Formally, consider a graph $G = (V, E)$ embedded in the plane:

- An edge crossing happens if two edges $e_1 = (u, v)$ and $e_2 = (x, y)$ intersect at a point p, where p is not one of the vertices u, v, x, or y.

If a graph can be drawn without any edge crossings, it is known as a planar graph.

Definition 18. (cf.[178, 456, 524]) A graph $G = (V, E)$ is called a *planar graph* if it can be drawn on a plane in such a way that no two edges intersect except at their endpoints. In other words, there exists a way to represent the graph on a two-dimensional plane such that the edges only meet at shared vertices, without any of the edges crossing each other.

Example 19 (Example of a Planar Graph)**.** One of the simplest examples of a planar graph is the complete graph K_4 minus one edge.

Consider the graph $G = (V, E)$ with the vertex set $V = \{1, 2, 3, 4\}$ and the edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{3, 4\}\}.$$

This graph G consists of 4 vertices and 5 edges connecting them.

The graph can be drawn on a plane in such a way that none of its edges intersect except at their endpoints. A common drawing of this graph resembles a triangle with an additional vertex inside connected to all three vertices of the triangle.

Since there exists a way to draw this graph on a two-dimensional plane without any edges crossing each other, it serves as an example of a planar graph.

Definition 20. (cf.[202, 458]) A graph $G = (V, E)$ is called a *non-planar graph* if it cannot be drawn on a plane without edge intersections, except at their endpoints. In other words, there is no way to represent G on a two-dimensional plane such that all edges meet only at shared vertices without any edges crossing each other.

Equivalently, a graph G is non-planar if every possible drawing of G in the plane results in at least one pair of edges intersecting at a point other than their endpoints.

Example 21 (Example of a Non-Planar Graph)**.** A classic example of a non-planar graph is the complete graph K_5.

Consider the graph $G = (V, E)$ with the vertex set

$$V = \{1, 2, 3, 4, 5\}$$

and the edge set containing all possible edges between these 5 vertices:

$$E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}.$$

In total, this graph has 5 vertices and 10 edges, with each pair of vertices connected by a unique edge.

No matter how we attempt to draw K_5 on a plane, there will always be at least one pair of edges that intersect at a point other than their endpoints. Therefore, it is impossible to draw K_5 in a way that avoids edge crossings.

Hence, K_5 is an example of a non-planar graph.

Among the many graphs related to Planar Graphs, one well-known example is the co-planar graph. The definition is provided below.

Definition 22. (cf.[163, 232, 345]) A graph $G = (V, E)$ is called a *co-planar graph* if its complement graph $\overline{G}$, defined as the graph having the same set of vertices V but with edges between vertices that are not connected in G, is a planar graph. In other words, G is co-planar if $\overline{G}$ can be drawn on a plane such that no two edges intersect except at their endpoints.

Example 23 (Example of a Co-Planar Graph)**.** Consider the graph $G = (V, E)$ with vertex set

$$V = \{1, 2, 3, 4\}$$

and edge set

$$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

The complement graph $\overline{G}$ is constructed by taking the same set of vertices V but adding edges between every pair of vertices that are not connected in G. Therefore, the edge set of $\overline{G}$ becomes:

$$E(\overline{G}) = \{\{1, 4\}\}.$$

The complement graph $\overline{G}$ has only one edge and can be drawn on a plane without any edge intersections. As a result, $\overline{G}$ is a planar graph.

Hence, G is an example of a co-planar graph.

We provide the definition of a k-planar graph [55, 59, 100], which generalizes the concept of a planar graph. Notably, related graph classes such as 1-planar graphs [292, 515] and 2-planar graphs [88, 227] are well-known.

Definition 24. (cf.[55, 59, 100]) A graph $G = (V, E)$ is called a *k-planar graph* if there exists a drawing of G in the plane such that each edge $e \in E$ is crossed by at most k other edges.

Formally, let D be a drawing of G on a plane where:

- V is the set of vertices in G.
- E is the set of edges, where each edge $e = (u, v)$ for $u, v \in V$ is represented as a continuous curve connecting u and v.

The graph G is said to be k-planar if, for every edge $e \in E$, the number of intersection points with other edges in the drawing D does not exceed k. That is, for every $e = (u, v) \in E$, there exists a drawing D such that

$$\text{cross}(e) \leq k,$$

where $\text{cross}(e)$ denotes the number of times edge e intersects with other edges in D.

In other words, a k-planar graph is a graph that can be drawn in the plane such that no edge is crossed by more than k other edges. For instance:

- If $k = 0$, the graph is a *planar graph*, meaning no edges cross.
- If $k = 1$, the graph is a 1-planar graph, meaning each edge can be crossed at most once.

The planar graph is a manageable and uncomplicated graph with no intersecting edges, making it suitable for various applications. Some examples of applications are provided below.

- urban street patterns: Urban street patterns refer to the layout and organization of streets in a city, including grid, radial, and irregular designs[82, 380, 390]. This has also been studied in the context of planar graphs [136, 337, 381].
- neural network: A neural network is a computational model inspired by the human brain, used for pattern recognition and data analysis[5, 277]. In the field of graphs, graph neural networks are frequently studied[219, 475, 550, 552, 563]. This has also been studied in the context of planar graphs [3, 197, 428].
- Computer vision: Computer vision is a field of artificial intelligence that enables computers to interpret and understand visual information from images or videos [220, 228, 537]. This has also been studied in the context of planar graphs [412, 562].
- VLSI: VLSI (Very Large Scale Integration) is the process of creating integrated circuits by combining thousands of transistors into a single chip [384, 481, 548]. In the VLSI field, VLSI floorplanning is often studied. VLSI floorplanning determines the size, shape, and placement of modules in a chip. It is frequently researched from a graph perspective since it helps estimate the chip area, interconnects, and delay[152, 223, 534]. This has also been studied in the context of planar graphs [38, 160, 343, 529, 530].
- Graph drawing: Graph drawing is a field that creates visual representations of graphs, helping to analyze relationships in data, networks, and applications such as social network analysis, linguistics, and bioinformatics[157, 317, 520]. Many papers have also been published on the graph drawing of planar graphs[295, 419].

2.2.2 Outerplanar graph

The outerplanar graph, like the planar graph, has been the subject of extensive research. Some related graph parameters include outerthickness[104], Outercoarseness[104], outerplanarity[99, 141, 312], and outer-2-planarity[293, 561]. Additionally, similar to planar graphs, numerous studies have focused on the applications of outerplanar graphs in networks and other fields. The definitions of outerplanar graphs, as well as their generalized concepts of outer-k-planar graphs and k-outerplanar graphs, are provided below.

Definition 25. (cf.[273, 409]) An *outer-planar graph* $G = (V, E)$ is an undirected graph that can be embedded in the plane such that all vertices lie on the outer face of the embedding, meaning no vertex is entirely enclosed by edges. In other words, it is possible to draw the graph on a plane without any edge crossings, with all vertices positioned on the unbounded (outer) face of the graph.

1. *Edge Crossing Condition*: The graph G must be drawn in such a way that no two edges intersect except at their endpoints.
2. *Outer Face Requirement*: All vertices of G must lie on the boundary of a single, unbounded face of the graph.

Definition 26. (cf.[224]) An *outer-k-planar graph* is a graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and each edge is crossed at most k times by other edges.

Formally, let $G = (V, E)$ be a graph, and consider a drawing D of G in the plane such that:

- All vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D.
- Each edge $e \in E$ is represented as a curve connecting its endpoints.

The graph G is called *outer-k-planar* if there exists a drawing D such that any edge $e \in E$ is crossed by at most k other edges. That is, for every edge $e \in E$,

$$\text{cross}(e) \leq k,$$

where $\text{cross}(e)$ denotes the number of times edge e intersects with other edges in the drawing D.

Definition 27. (cf.[58, 97, 143, 318]) A graph $G = (V, E)$ is called a *k-outerplanar graph* if it has a planar embedding such that the vertices belong to at most k concentric layers with respect to the outer face of the graph.

Formally, a graph G is said to be k-outerplanar if there exists a planar embedding D of G in which the vertices can be partitioned into k layers as follows:

1. The *first layer* consists of all vertices that lie on the outer face (unbounded face) of the embedding D, making this subgraph an outerplanar graph.
2. For $i > 1$, the *i-th layer* is defined recursively by removing all vertices that belong to layers 1 through $i - 1$. The remaining vertices that lie on the unbounded face of the resulting subgraph form the i-th layer.
3. The process stops when all vertices are assigned to one of the k layers.

If such a layering exists for the graph G, then G is k-outerplanar.

The following theorem holds.

Theorem 28. *(cf.[13, 126]) Every outerplanar graph is a planar graph.*

The related graph classes within the planar graph category are listed below. The numerous research studies on these related classes highlight the importance of studying planar graphs.

Notation 29. In this paper, we define the term ”Related graph class” as a graph class that either extends or restricts a corresponding graph class in some way.

Theorem 30. *The following are examples of related graph classes, including but not limited to:*

- *Halin graphs [113]*
- *Fan-planar graphs [326]*
- *Fan-crossing free graphs[123]*
- *IC-planar graphs[125]*
- *Upward planar graphs [16, 74, 316]*
- *Convex planar graphs [187]*
- *1-planar graphs [292, 515]*
- *2-planar graphs [88, 227]*
- *maximal 2-planar graphs[290]*
- *3-planar graphs [87]*
- *Interval Planar Graphs [315]*
- *k-Planar Graphs [84]*
- *Outer-1-planar graphs [75, 98, 362]*

- *Outer-k-planar graphs [141]*
- *Quasi-planar graphs [8, 12]*
- *k-Quasi-planar Graphs [231, 509, 510]*
- *Apex graphs [158, 395, 396, 422]* [3]
- *Pseudo-outerplanar graphs[526]*
- *biconnected outerplanar graphs[140, 369]*
- *Strongly-connected outerplanar graphs[230]*
- *Biconnected planar graphs[105]*
- *Gap-planar graphs [79]*
- *Non-separating planar graphs[181]*
- *NIC-planar graphs[78, 421, 525]*
- *Apex-Outerplanar Graphs[199]*
- *Universal outerplanar graphs[370]*
- *Outer-projective-planar graphs[65]*
- *Planar Ramsey graphs[76]*
- *Cubic outerplanar graphs[269]*
- *Outerplanar partial cubes graphs[461]*
- *Directed Planar graphs[118, 323, 341]*
- *Planar partial cubes graphs[183, 439]*
- *Planar median graphs[479]*
- *Planar straight-line graph[190]*
- *Directed outerplanar graphs[138]*
- *Maximum Planar Subgraph[134]*
- *Weighted planar graph[93, 94]*
- *RAC Graph[57, 191, 192]*
- *Planarly-connected graphs [7]*
- *2-connected planar graphs[389]*
- *Bipartite IC-planar graphs[57]*
- *Bipartite NIC-planar graphs[57]*
- *Bipartite RAC Graphs[57]*
- *Cubic planar graph[403]*
- *Regular planar graphs[89, 416, 551]*
- *Partitioning planar graphs[149]*
- *locally planar graphs[282]*
- *Bipartite Planar Graphs[226]*
- *Signed planar graphs[411]*
- *Random cubic planar graphs[45]*
- *Random planar graphs[259, 383, 427]*
- *Random outerplanar graphs[106, 322]*

Proof. Refer to each reference as needed. □

[3] The term ”apex” is sometimes used in contexts beyond planar graphs to indicate how close a graph is to a desired graph structure [355].

2.2.3 Apex graph

An apex graph becomes planar by removing one vertex; it is non-planar but can achieve planarity this way. It has been extensively studied alongside planar graphs [158, 395, 396, 422]. It is closely related to the operation of vertex deletion (cf.[248, 310])in graphs. The definition is provided below.

Definition 31. (cf.[158, 395, 396, 422]) An *apex graph* is a graph $G = (V, E)$ that can be made planar by the removal of a single vertex $v \in V$. The removed vertex is called an *apex* of the graph G.

Formally, a graph G is an apex graph if there exists a vertex $v \in V$ such that the subgraph $G' = G[V \setminus \{v\}]$, obtained by removing v and all edges incident to v, is a planar graph.

The properties of an apex graph include:

- If the graph G itself is planar, then every vertex can be considered an apex.
- Apex graphs may have more than one vertex that can serve as an apex, and if multiple such vertices exist, each can be independently removed to obtain a planar graph.
- The null graph (i.e., a graph with no vertices or edges) is trivially considered an apex graph.

The k-apex graph, a generalization of the aforementioned apex graph, has also been extensively studied. The definition is provided below.

Definition 32. (cf.[161,214,263]) A *k-apex graph* is a graph $G = (V, E)$ that can be made planar by the removal of at most k vertices from V. In other words, there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is a planar graph.

The properties of a k-apex graph include:

- If $k = 0$, the k-apex graph itself is already a planar graph.
- When $k = 1$, the k-apex graph corresponds to the definition of an apex graph, where the removal of a single vertex makes the graph planar.
- A k-apex graph may have multiple different subsets of vertices S that can be removed to achieve planarity.

2.2.4 Apex outerplanar graph

An apex outerplanar graph becomes planar by removing one vertex. It has been extensively studied alongside outerplanar graphs [200, 209]. The definition is provided below.

Definition 33. [200, 209]. An *apex outerplanar graph* is a graph $G = (V, E)$ that can be made outerplanar by the removal of a single vertex $v \in V$. The removed vertex v is called an *apex* of the outerplanar graph G.

Formally, G is an apex outerplanar graph if there exists a vertex $v \in V$ such that the subgraph $G' = G[V \setminus \{v\}]$, obtained by removing v and all edges incident to v, is an outerplanar graph.

The properties of an apex outerplanar graph include:

- If the graph G itself is outerplanar, then every vertex in V can be considered an apex.
- An apex outerplanar graph may have more than one vertex that can serve as an apex, and if multiple such vertices exist, each can be independently removed to obtain an outerplanar graph.
- The null graph (i.e., a graph with no vertices or edges) is trivially considered an apex outerplanar graph.

Definition 34. A graph $G = (V, E)$ is called a *k-apex outerplanar graph* if it can be made outerplanar by the removal of at most k vertices from V.

Formally, G is a k-apex outerplanar graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is an outerplanar graph.

The properties of a k-apex outerplanar graph include:

- If $k = 0$, the k-apex outerplanar graph itself is already outerplanar.
- When $k = 1$, the graph is termed an *apex outerplanar graph*, meaning there exists a single vertex v whose removal results in an outerplanar graph.
- For general k, multiple subsets S of vertices may exist, such that their removal results in an outerplanar graph, indicating that the graph G has more flexibility in achieving outerplanarity compared to a standard outerplanar graph.

The relationships between planar graphs, outerplanar graphs, apex graphs, and apex outerplanar graphs are illustrated in the following diagram.

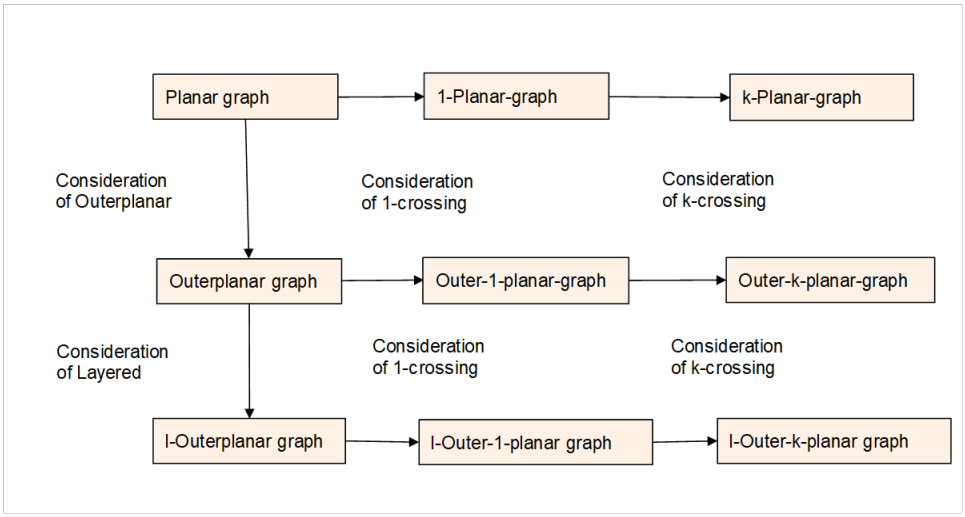

Fig. 2. Graph Hierarchy for Planar graph

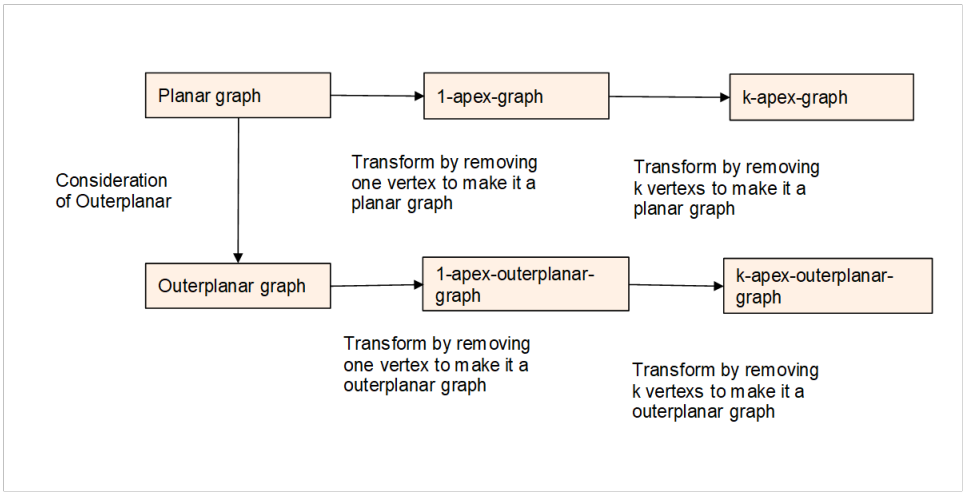
Planar graph
1-apex-graph
k-apex-graph
Consideration of Outerplanar
Transform by removing one vertex to make it a planar graph
Transform by removing k vertexs to make it a planar graph
Outerplanar graph
1-apex-outerplanar-graph
k-apex-outerplanar-graph
Transform by removing one vertex to make it a outerplanar graph
Transform by removing k vertexs to make it a outerplanar graph

Fig. 3. Graph Hierarchy for Apex graph

2.2.5 Quasi-planar graph

A quasi-planar graph is a graph that can be drawn in the plane such that no three edges intersect each other pairwise [4]. Quasi-planar graphs have been extensively studied alongside planar graphs [8, 12, 60, 122, 188, 189, 344]. The definition is provided below.

Definition 35. [8, 12] A graph $G = (V, E)$ is called a *quasi-planar graph* if it can be drawn in the plane such that no three edges are pairwise crossing. In other words, there is no set of three edges in G that intersect each other at distinct points in their interiors.

The properties of a quasi-planar graph include:

- Every planar graph is inherently quasi-planar because planar graphs have no edge crossings.
- Quasi-planar graphs have been conjectured to have at most $O(n)$ edges for n vertices.

[4] In this context, "pairwise" means that no two edges within any group of three edges intersect each other. Therefore, a quasi-planar graph is one where any set of three edges has no two edges that intersect each other at the same time.

An example of a graph that extends the concept of a quasi-planar graph is the k-quasi-planar graph [56, 231, 509]. The definition is provided below.

Definition 36. [56, 231] A graph $G = (V, E)$ is called a *k-quasi-planar graph* if it can be drawn in the plane such that no k edges are pairwise crossing. In other words, there does not exist a subset of k edges in G that all intersect each other at distinct points in their interiors.

The properties of a k-quasi-planar graph include:

- For $k = 2$, a k-quasi-planar graph is simply a planar graph because no two edges cross each other.
- For any fixed $k \geq 2$, a k-quasi-planar graph on n vertices has been conjectured to have at most $O(n)$ edges.
- k-quasi-planar graphs generalize the concept of quasi-planar graphs as they permit more pairwise crossings while maintaining an upper bound on the crossing behavior.

2.2.6 outer quasi planar graph

We define the *outer k-quasi-planar graph* as a related graph class to the ones mentioned above. An outer quasi-planar graph is a graph that combines the concepts of an outerplanar graph and a quasi-planar graph.

Definition 37. An *outer-k-quasi-planar graph* is a graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k edges are pairwise crossing.

Formally, let $G = (V, E)$ be a graph, and consider a drawing D of G in the plane such that:

- All vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D.
- Each edge $e \in E$ is represented as a curve connecting its endpoints.

The graph G is called *outer-k-quasi-planar* if there exists a drawing D such that any set of k edges does not intersect each other at distinct points in their interiors.

2.2.7 quasi apex graph

We define the quasi apex graph as related graph classes to the ones mentioned above.

Definition 38. A *k-quasi apex graph* is a graph $G = (V, E)$ that can be made planar by the removal of at most k vertices.

Formally, G is a k-quasi apex graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is a planar graph.

The properties of a k-quasi apex graph include:

- If $k = 0$, the k-quasi apex graph itself is already planar.
- When $k = 1$, the graph corresponds to an apex graph where the removal of a single vertex makes the graph planar.
- The k-quasi apex graph generalizes the concept of apex graphs by allowing up to k vertices to be removed to achieve planarity.

2.2.8 quasi apex outerplanar graph

We define the quasi apex outerplanar graph as related graph classes to the ones mentioned above.

Definition 39. A *k-quasi apex outerplanar graph* is a graph $G = (V, E)$ that can be made outerplanar by the removal of at most k vertices.

Formally, G is a k-quasi apex outerplanar graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced subgraph $G' = G[V \setminus S]$, obtained by removing all vertices in S along with their incident edges, is an outerplanar graph.

The properties of a k-quasi apex outerplanar graph include:

- If $k = 0$, the k-quasi apex outerplanar graph itself is already outerplanar.
- When $k = 1$, the graph is termed an apex outerplanar graph, meaning there exists a single vertex v whose removal results in an outerplanar graph.
- For general k, multiple subsets S of vertices may exist such that their removal results in an outerplanar graph, indicating that the graph G has more flexibility in achieving outerplanarity compared to a standard outerplanar graph.

The relationship diagram between quasi-planar graphs is shown below.

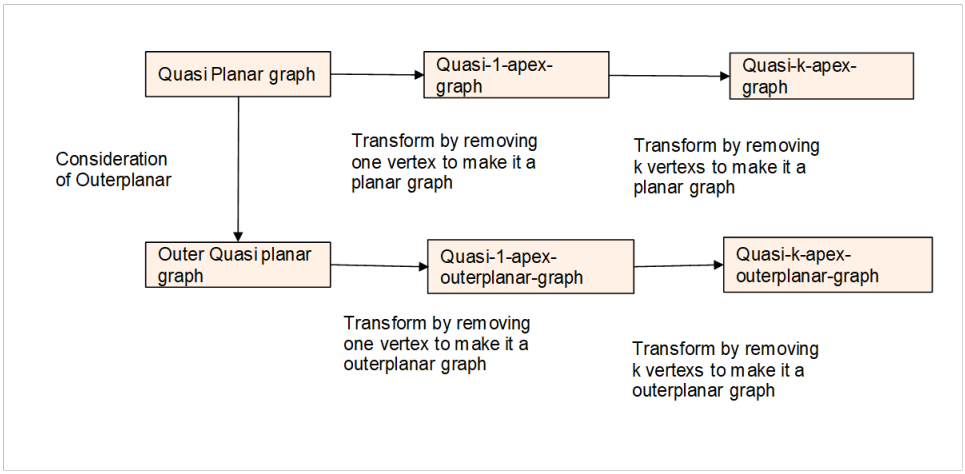
Quasi Planar graph
Quasi-1-apex-graph
Quasi-k-apex-graph
Consideration of Outerplanar
Transform by removing one vertex to make it a planar graph
Transform by removing k vertexs to make it a planar graph
Outer Quasi planar graph
Quasi-1-apex-outerplanar-graph
Quasi-k-apex-outerplanar-graph
Transform by removing one vertex to make it a outerplanar graph
Transform by removing k vertexs to make it a outerplanar graph

Fig. 4. Graph Hierarchy for Quasi-Planar graph

2.2.9 Gap-planar graph and Fan-planar graph

Graphs related to planar graphs include gap-planar graphs [79,477] and fan-planar graphs [86, 101, 326, 366]. Several research papers have been published on these topics. Their definitions are provided below.

Definition 40. [79] A graph $G = (V, E)$ is called a *k-gap-planar graph* if it admits a drawing in the plane such that each crossing is assigned to one of the two involved edges, and each edge is assigned at most k of its crossings. In other words, in the drawing of G, each edge can be crossed by at most k other edges, where each crossing is uniquely associated with one of the edges involved in the crossing.

The properties of a k-gap-planar graph include:

- When $k = 0$, a k-gap-planar graph is simply a planar graph with no crossings.
- k-gap-planar graphs provide a way to measure how far a graph is from being planar, by limiting the number of crossings assigned to each edge.

Definition 41. (cf.[86, 101, 326, 366]) A graph $G = (V, E)$ is called a *fan-planar graph* if it admits a drawing in the plane such that every edge is crossed only by a set of pairwise adjacent edges (forming what is known as a ”fan”). This means that any edge e in G can only be crossed by edges that share a common vertex with each other, forming a fan-like pattern.

The properties of a fan-planar graph include:

- The fan-planar condition restricts how edges can intersect, allowing a more structured way of crossing, unlike general non-planar graphs.
- If a graph is fan-planar, it means there exists a drawing where the crossings exhibit a ”fan-like” behavior.

And Graphs related to outerplanar graphs include outer-fan-planar graph. Several research papers have been published on these topics [85, 102, 123, 124]. Their definitions are provided below.

Definition 42. (cf.[85, 102, 123, 124]) A graph $G = (V, E)$ is called an *outer-fan-planar graph* if it has a planar embedding in which all vertices lie on the boundary of the unbounded (outer) face, and every edge is crossed only by pairwise adjacent edges, known as *fans*.

More formally:

- G can be drawn in the plane such that all vertices are positioned on a single circle C, which represents the outer face.
- The embedding must be such that no two edges cross more than once, and if an edge e is crossed by other edges, all those crossing edges must share a common vertex incident to e.

An important property of outer-fan-planar graphs is that they are *biconnected*, meaning they remain connected after the removal of any single vertex. This is necessary for a graph to be maximal outer-fan-planar.

For example:

- If the graph G contains a cut-vertex c (a vertex whose removal disconnects the graph), then it is possible to add an additional edge between two neighbors of c without violating outer-fan-planarity, thus indicating that G was not maximal outer-fan-planar.
- A biconnected graph G is outer-fan-planar if and only if it admits a straight-line outer-fan-planar drawing where the vertices are restricted to lie on a circle C, preserving their cyclic order in this embedding.

It has been proven that any outer-fan-planar graph with n vertices has at most $5n - 10$ edges.

2.2.10 Inner planar graph

Recently, the concept of an Inner Planar Graph has been defined. The definition is provided below[426].

Definition 43. [426] A graph $G = (V, E)$ is called a *k-inner planar graph* if it has a planar embedding with at most k vertices that do not lie on the boundary of the outer face. These vertices are referred to as *inner vertices*.

More formally:

- G is a planar graph that can be drawn on the plane such that when embedded, at most k vertices do not lie on the boundary of the outer (unbounded) face of the graph.
- The remaining vertices must lie on the boundary of the outer face.

Special cases of k-inner planar graphs include:

- When $k = 0$, the graph is called an *outerplanar graph*, meaning all vertices lie on the boundary of the outer face.
- For $k > 0$, the graph allows up to k vertices to be positioned inside the outer face, while still maintaining a planar embedding.

2.2.11 Almost Planar Graph

The concept of an Apex Graph focuses on vertices, whereas graphs centered on edges include the Almost Planar Graph[201,274,289] and Almost Outerplanar Graph[213,538]. These graphs are also referred to as Nearly Planar Graphs[216, 268] and Nearly Outerplanar Graphs, respectively. This graph class is closely related to the graph operation known as edge deletion (cf.[67, 485, 556, 557]). The definitions are introduced below.

Definition 44. [201,274,289] A graph G is called an *almost planar graph* if there exists an edge $e \in E(G)$ such that the graph $G - e$ (i.e., the graph obtained by removing the edge e from G) is planar.

Definition 45. (cf.[169, 170]) A graph $G = (V, E)$ is called a *k-almost planar graph* if there exists a subset $F \subseteq E(G)$ with $|F| \leq k$ such that the graph $G - F$ (i.e., the graph obtained by removing all edges in F from G) is planar.

In other words, a k-almost planar graph can be made planar by removing at most k edges.

Definition 46. [213, 538] A graph G is called an *almost outerplanar graph* if there exists an edge $e \in E(G)$ such that the graph $G - e$ (i.e., the graph obtained by removing the edge e from G) is outerplanar.

Definition 47. A graph $G = (V, E)$ is called a *k-almost outerplanar graph* if there exists a subset $F \subseteq E(G)$ with $|F| \leq k$ such that the graph $G - F$ (i.e., the graph obtained by removing all edges in F from G) is outerplanar.

In other words, a k-almost outerplanar graph can be made outerplanar by removing at most k edges.

Almost planar graphs and almost outerplanar graphs are related to the concept of *skewness-k drawings* in the field of graph drawing [156, 193]. A graph is said to have *skewness k* if it can be drawn with k edge deletions needed to make it planar[193].

In addition to the graph concepts described in the subsection, related notions such as (k, l)-grid-free drawings[429] and k-fan-crossing-free drawings[153] are also well-known. This indicates that research on planar graphs is considered highly valuable.

2.3 Fuzzy planar graph and Fuzzy outerplanar graph

In recent years, inspired by planar and outerplanar graphs, the concepts of fuzzy planar graphs and fuzzy outerplanar graphs have been introduced. To begin, the definitions of fuzzy graphs and fuzzy planar graphs are provided below. A fuzzy graph is an extension of graph theory that incorporates the principles of fuzzy sets [51, 135, 165, 179, 255, 371, 523, 553, 558, 559]. Extensive research has been conducted on both fuzzy graphs [96, 175, 260, 400, 452, 459, 505] and fuzzy planar graphs [20, 256, 441, 442, 472].

Definition 48. [459] A *fuzzy graph* $\psi = (V, \sigma, \mu)$ is defined as follows:

- V is a set of vertices.
- $\sigma : V \to [0, 1]$ is a function that assigns a membership degree to each vertex $v \in V$, indicating the degree of membership of v in the fuzzy graph.
- $\mu : V \times V \to [0, 1]$ is a fuzzy relation that represents the strength of the connection between each pair of vertices $(u, v) \in V \times V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

In this definition, the following properties hold:

- The fuzzy function μ is symmetric, meaning $\mu(u, v) = \mu(v, u)$ for all $u, v \in V$.
- Additionally, $\mu(v, v) = 0$ for all $v \in V$, meaning that there is no self-loop in the fuzzy graph.

The fuzzy graph ψ allows for the representation of uncertainty in the presence or strength of connections between vertices, making it a valuable tool for modeling complex systems with ambiguous or imprecise relationships.

Theorem 49. *[235] The following are examples of related graph classes, including but not limited to:*

- *Bipolar Fuzzy Graphs [17]*
- *Fuzzy Planar Graphs [470]*
- *Complex Hesitant Fuzzy Graph [6]*
- *Constant hesitancy fuzzy graphs[434]*
- *Bipolar hesitancy fuzzy graph[299, 430]*
- *Hesitancy fuzzy magic labeling graph[450]*
- *Dual hesitant fuzzy graphs[80]*
- *Regular Hesitancy Fuzzy Soft Graphs[352]*
- *Hesitant fuzzy hypergraphs[265]*
- *Irregular Bipolar Fuzzy Graphs [469]*
- *Regular Bipolar Fuzzy Graphs [24]*
- *Picture Fuzzy Tolerance Graphs [172]*
- *Complex Hesitant Fuzzy Graphs [6]*
- *Strong Intuitionistic Fuzzy Graphs [23]*
- *Product Fuzzy Graphs [41]*
- *Partially Total Fuzzy Graphs [11]*
- *Fuzzy Influence Graphs [382]*
- *Picture Fuzzy Directed Hypergraphs [333]*
- *Radio Fuzzy Graphs [374]*
- *Line Regular Fuzzy Semigraphs [64]*
- *Fuzzy Incidence Graphs [198]*
- *Balanced Picture Fuzzy Graphs [53]*
- *Oscillating Polar Fuzzy Graphs [52]*
- *Cayley Fuzzy Graphs [519]*
- *Rough Fuzzy Digraphs [14]*
- *T-Spherical Fuzzy Graphs [275]*

- *Mixed Fuzzy Graphs [465]*
- *Einstein Fuzzy Graphs [308]*
- *Edge-Regular Fuzzy Graphs [42, 137, 425]*
- *Robust Fuzzy Graphs [505]*
- *Anti-Product Fuzzy Graphs [39]*
- *Valued Fuzzy Superhypergraphs [494]*
- *Inverse Fuzzy Graphs [120]*
- *Inverse Eccentric Fuzzy Graphs [385]*
- *Cubic Pythagorean Fuzzy Graphs [403]*
- *Complete Fuzzy Graphs [40]*
- *Mixed Picture Fuzzy Graphs [407]*
- *Extended Total Fuzzy Graphs [10]*
- *Pseudo Regular Fuzzy Graphs [377]*
- *Best Fuzzy Graphs [512]*
- *Intuitionistic Felicitous Fuzzy Graphs [95]*
- *Middle Fuzzy Graphs [392, 507]*
- *Bipolar Fuzzy P-Competition Graphs [420]*
- *Fuzzy Intersection Graphs [448]*
- *Fuzzy Semigraphs [462]*
- *Intuitionistic Fuzzy Soft Expert Graphs [533]*
- *m-Polar Fuzzy Graphs [19]*
- *Balanced Interval-Valued Fuzzy Graphs [446]*
- *Double Layered Fuzzy graph[338]*
- *Triple Layered Fuzzy Graph[460]*
- *Fuzzy Outerplanar Graphs [309]*
- *Inverse fuzzy multigraphs[119]*
- *Bipolar inverse fuzzy graphs[332]*
- *Fuzzy zero divisor graphs[348, 349]*
- *General Fuzzy graphs[234, 417]*
- *T-Spherical Fuzzy Graphs[275]*
- *Spherical Fuzzy Labelling Graphs[145]*
- *spherical fuzzy digraphs[379]*
- *T-spherical fuzzy Hamacher graphs[68]*
- *Regular spherical fuzzy graph[393]*
- *Spherical Fuzzy Cycle graph[353]*
- *Spherical Fuzzy Tree graph[353]*
- *Pseudo regular spherical fuzzy graphs[284]*
- *Cubic Pythagorean fuzzy graphs [403]*
- *Complex Pythagorean Dombi fuzzy graphs [26]*
- *Interval-valued Pythagorean fuzzy graphs[29]*
- *Interval-valued complex Pythagorean fuzzy graph[499]*
- *Pythagorean Dombi fuzzy graphs[22]*
- *Pythagorean fuzzy soft graphs[482]*
- *Pythagorean neutrosophic fuzzy graphs[15]*
- *Complex pythagorean fuzzy planar graphs[20]*

- *Pythagorean neutrosophic Dombi fuzzy graphs [144]*
- *Complex Pythagorean fuzzy threshold graphs[18]*
- *Pythagorean Dombi fuzzy soft graphs[34]*
- *Pythagorean fuzzy incidence graphs[36, 484]*
- *Picture Dombi Fuzzy Graph [394]*
- *Picture fuzzy line graphs[151]*
- *Picture fuzzy planar graphs[92]*
- *Picture fuzzy tolerance graphs[172]*
- *Picture fuzzy ϕ-tolerance competition graphs[173]*
- *Interval-Valued Picture Fuzzy Graph [445]*
- *Picture Fuzzy Incidence Graph [413]*
- *Picture fuzzy threshold graphs[172]*
- *Picture Fuzzy Soft Graph [147]*
- *Cayley Picture Fuzzy Graph [334]*
- *m-Polar Picture Fuzzy Graphs[335]*
- *Interval-Valued Picture (S, T)-Fuzzy Graph [66]*
- *q-Rung Picture Fuzzy Graph [25]*
- *Mixed Picture Fuzzy Graph [407]*
- *Picture Fuzzy Directed Hypergraphs[333]*
- *Picture fuzzy cubic graphs[336]*
- *Picture Fuzzy Digraph [379]*
- *Balanced Picture Fuzzy Graph [54]*

Considering these fuzzy graph classes enables the identification of shared properties, which can lead to the development of efficient algorithms, deeper analysis, and practical applications across various fields.

Proof. Refer to each reference as needed. □

Definition 50. (cf.[20, 256, 441, 442, 472]) A fuzzy graph $\psi = (V, \sigma, \mu, E)$ is termed a *fuzzy planar graph* if it consists of:

- A set of vertices V, where each vertex $v \in V$ is associated with a membership degree through a fuzzy set $\sigma : V \to [0, 1]$.
- A fuzzy relation $\mu : V \times V \to [0, 1]$ representing the strength of the connection between pairs of vertices, where $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$ holds for all $u, v \in V$.

  Additionally,

- The edge set E is represented by a fuzzy set on the Cartesian product of V, i.e., $E \subseteq V \times V$, with each edge $(u, v) \in E$ having a membership value $\mu(u, v)$ indicating the degree to which (u, v) belongs to E.
- The fuzzy graph ψ is said to be planar if it can be drawn on a plane without any intersecting edges, adhering to the fuzziness in the representation. Specifically, if two fuzzy edges (a, b) and (c, d) intersect, there exists an intersection point p, and the intersection strength is defined as

$$I_p = \frac{I(a, b) + I(c, d)}{2},$$

  where

$$I(a, b) = \frac{\mu(a, b)}{\sigma(a) \wedge \sigma(b)}.$$

The fuzzy planarity value f measures how planar the graph is and is given by

$$f = \frac{1}{1 + \sum_{i=1}^{n} I_{p_i}}.$$

For a completely planar fuzzy graph, $f = 1$.

Proposition 51. *A fuzzy planar graph is a fuzzy graph.*

Proof. Obviously holds. □

Theorem 52. *The following are examples of related graph classes for fuzzy planar graphs, including but not limited to:*

- *Interval-valued fuzzy planar graphs[442]*
- *m-polar fuzzy planar graphs[257, 258]*
- *Complex pythagorean fuzzy planar graphs[21]*
- *Complex q□Rung Orthopair Fuzzy Planar Graphs[298]*
- *Picture fuzzy planar graphs[92]*
- *Inverse Fuzzy Mixed Planar Graphs[398]*
- *Bipolar Fuzzy Planar Graphs[30]*

Proof. Refer to each reference as needed. □

2.3.1 Fuzzy outerplanar graph

The definitions of fuzzy outerplanar graphs are provided below [309]. These graphs incorporate the concepts of fuzzy graphs into the structure of outerplanar graphs.

Definition 53. [309] A fuzzy graph $\psi = (V, \sigma, \mu)$ qualifies as a *fuzzy outerplanar graph* if:

- It can be embedded in a plane such that every vertex $v \in V$ lies on the boundary of the exterior region of the embedding.

More specifically:

- Similar to the fuzzy planar graph, $\sigma : V \to [0, 1]$ is a fuzzy set indicating the degree of membership of each vertex, and $\mu : V \times V \to [0, 1]$ indicates the strength of the connection between vertices.
- For a fuzzy graph to be outerplanar, there must be no fuzzy edge intersection while ensuring all vertices are positioned on the outer face.

If $i(\psi)$ represents the number of vertices not on the boundary of the outer face, then a fuzzy outerplanar graph satisfies $i(\psi) = 0$. In contrast, if $i(\psi) \neq 0$, the graph is considered a fuzzy non-outerplanar graph.

These definitions extend the classical notions of planarity and outerplanarity into the fuzzy domain, allowing for the representation and analysis of uncertainty in graph structures.

Proposition 54. *A fuzzy outerplanar graph is a fuzzy graph.*

Proof. Obviously holds. □

Proposition 55. *A fuzzy outerplanar graph $\psi = (V, \sigma, \mu)$ can be transformed into a classical outerplanar graph $G = (V, E)$ by considering only those edges $(u, v) \in E$ where the membership degree $\mu(u, v)$ is strictly positive.*

Proof. The transformation process is as follows:

1. *Vertex Set*: Retain the original set of vertices V from ψ.
2. *Edge Set*: Construct an edge set E for the outerplanar graph $G = (V, E)$ such that $(u, v) \in E$ if and only if $\mu(u, v) > 0$ in the fuzzy graph ψ.

Since ψ is a fuzzy outerplanar graph, it satisfies the condition that all vertices lie on the boundary of the exterior region, and there are no fuzzy edge intersections in the embedding. This means that the resulting graph $G = (V, E)$ will be outerplanar, as it maintains the same embedding properties:

- *No edge crossings* occur in G, as all vertices and edges inherited from ψ satisfy the outerplanarity condition.
- *All vertices remain on the boundary* of the outer face, as required by the definition of an outerplanar graph.

Thus, by disregarding the fuzziness (i.e., ignoring the exact membership values $\sigma(v)$ and $\mu(u, v)$), the fuzzy outerplanar graph ψ is transformed into a classical outerplanar graph G.

This completes the transformation, proving that every fuzzy outerplanar graph ψ can be reduced to a corresponding outerplanar graph G. □

2.3.2 Fuzzy apex graph

In general graph classes, a type of graph known as an apex graph is well-known. Similar to planar graphs, apex graphs have been the subject of extensive research [158, 395–397, 422]. Incorporating the concept of fuzzy relations, the definition of a fuzzy apex graph is provided below.

Definition 56. (cf.[158, 395–397, 422].) A *fuzzy apex graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy planar graph by the removal of a single vertex $v \in V$. The removed vertex is called an *apex* of the fuzzy graph ψ.

Formally, ψ is a fuzzy apex graph if there exists a vertex $v \in V$ such that the subgraph $\psi' = (V \setminus \{v\}, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ on the remaining vertices, is a fuzzy planar graph.

The fuzzy apex graph retains the following properties:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices in V, with $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.
- If the fuzzy apex graph ψ itself is fuzzy planar, then every vertex in V can be considered an apex.

Similar to classical apex graphs, a fuzzy apex graph may have multiple apex vertices. Additionally, if ψ is the null fuzzy graph (i.e., $V = \emptyset$), it is trivially a fuzzy apex graph.

Proposition 57. *A fuzzy apex graph can be transformed into a fuzzy planar graph.*

Proof. Obviously holds. □

Definition 58. A *fuzzy k-apex graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be transformed into a fuzzy planar graph by removing at most k vertices from V.

If $k = 0$, the fuzzy k-apex graph itself is already a fuzzy planar graph. When $k = 1$, the graph corresponds to a fuzzy apex graph as defined earlier.

Proposition 59. *A fuzzy k-apex graph can be transformed into a fuzzy planar graph.*

Proof. Obviously holds. □

Proposition 60. *A fuzzy k-apex graph can be transformed into a fuzzy apex graph.*

Proof. Obviously holds. □

2.3.3 Fuzzy apex outerplanar graph

The concept of a fuzzy apex outerplanar graph and a k-apex outerplanar graph is extended to the following graph by incorporating the idea of fuzzy graphs.

Definition 61. A *fuzzy apex outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be transformed into a fuzzy outerplanar graph by the removal of a single vertex $v \in V$. The removed vertex is called an *apex* of the fuzzy graph ψ.

Formally, ψ is a fuzzy apex outerplanar graph if there exists a vertex $v \in V$ such that the induced subgraph $\psi' = (V \setminus \{v\}, \sigma', \mu')$, where σ' and μ' are the restrictions of the membership function σ and the adjacency function μ on the remaining vertices, is a fuzzy outerplanar graph.

The fuzzy apex outerplanar graph retains the following properties:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices in V, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

- If the fuzzy apex outerplanar graph ψ itself is a fuzzy outerplanar graph, then every vertex in V can be considered an apex.

Proposition 62. *A fuzzy apex outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. Obviously holds. □

Definition 63. A *fuzzy k-apex outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy outerplanar graph by the removal of at most k vertices from V.

Formally, ψ is a fuzzy k-apex outerplanar graph if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced fuzzy subgraph $\psi' = (V \setminus S, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ on the remaining vertices, is a fuzzy outerplanar graph.

The properties of a fuzzy k-apex outerplanar graph include:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices in V, with $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.
- If $k = 0$, the fuzzy k-apex outerplanar graph itself is already a fuzzy outerplanar graph.
- When $k = 1$, the fuzzy k-apex outerplanar graph corresponds to a fuzzy apex outerplanar graph, where the removal of a single vertex makes the graph fuzzy outerplanar.
- For general k, multiple subsets S of vertices may exist such that their removal results in a fuzzy outerplanar graph, indicating that the graph ψ has more flexibility in achieving outerplanarity compared to a standard fuzzy outerplanar graph.

This generalization allows for incorporating fuzzy uncertainty into the structure of k-apex outerplanar graphs, providing a broader framework for analyzing graphs that have fuzzy relations and can become outerplanar by removing up to k vertices.

Theorem 64. *A fuzzy k-apex outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. Obviously holds. □

Theorem 65. *A fuzzy k-apex outerplanar graph can be transformed into a fuzzy apex outerplanar graph.*

Proof. Obviously holds. □

2.3.4 Fuzzy Quasi-Planar graph

The Fuzzy Quasi-Planar graph is an extension of the Quasi-Planar graph to a fuzzy graph. The definition of the Fuzzy Quasi-Planar graph is provided below.

Definition 66. A *fuzzy quasi-planar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be drawn in the plane such that no three edges are pairwise crossing.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ assigns a membership degree to each edge (u, v) representing the strength of the connection between vertices u and v, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called *fuzzy quasi-planar* if it can be drawn in such a way that any three fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 67. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy k-quasi-planar graph* if it can be drawn in the plane such that no k fuzzy edges are pairwise crossing. In other words, there does not exist a subset of k fuzzy edges in ψ that all intersect each other at distinct points in their interiors.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices $u, v \in V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called a *fuzzy k-quasi-planar graph* if there exists a drawing D in which any set of k fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 68. *A fuzzy k-quasi-planar graph can be transformed into a fuzzy quasi-planar graph.*

Proof. Obviously holds. □

2.3.5 Fuzzy Quasi-outerplanar graph

The Fuzzy Quasi-outerplanar graph is an extension of the Quasi-outerplanar graph to a fuzzy graph. The definition of the Fuzzy Quasi-outerplanar graph is provided below.

Definition 69. A *fuzzy quasi-outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three edges are pairwise crossing.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ assigns a membership degree to each edge (u, v) representing the strength of the connection between vertices u and v, satisfying $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called *fuzzy quasi-outerplanar* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any three fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 70. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy outer-k-quasi-planar graph* if it can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k fuzzy edges are pairwise crossing.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph where:

- $\sigma : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$.
- $\mu : V \times V \to [0, 1]$ represents the strength of the connection between pairs of vertices $u, v \in V$, such that $\mu(u, v) \leq \min\{\sigma(u), \sigma(v)\}$.

The fuzzy graph ψ is called *fuzzy outer-k-quasi-planar* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 71. *A fuzzy outer-k-quasi-planar graph can be transformed into a fuzzy outer-quasi-planar graph.*

Proof. Obviously holds. □

2.3.6 Fuzzy quasi apex graph

The Fuzzy quasi apex graph is an extension of the quasi apex graph to a fuzzy graph. The definition of the Fuzzy Quasi-outerplanar graph is provided below.

Definition 72. A *fuzzy k-quasi apex graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy quasi planar graph by removing at most k vertices.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph. The graph ψ is called a *fuzzy k-quasi apex graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced fuzzy subgraph $\psi' = (V \setminus S, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ to the remaining vertices, is a fuzzy planar graph.

The properties of a fuzzy k-quasi apex graph include:

- If $k = 0$, the fuzzy k-quasi apex graph itself is already a fuzzy quasi planar graph.
- When $k = 1$, the fuzzy graph corresponds to a fuzzy apex graph where the removal of a single vertex makes the graph fuzzy quasi planar.

Theorem 73. *A fuzzy k-quasi apex graph can be transformed into a fuzzy quasi-planar graph.*

Proof. Obviously holds. □

2.3.7 Fuzzy quasi apex outerplanar graph

The Fuzzy quasi apex outerplanar graph is an extension of the quasi apex outerplanar graph to a fuzzy graph. The definition of the Fuzzy apex outerplanar graph is provided below.

Definition 74. A *fuzzy k-quasi apex outerplanar graph* $\psi = (V, \sigma, \mu)$ is a fuzzy graph that can be made a fuzzy quasi outerplanar graph by removing at most k vertices.

Formally, let $\psi = (V, \sigma, \mu)$ be a fuzzy graph. The graph ψ is called a *fuzzy k-quasi apex outerplanar graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced fuzzy subgraph $\psi' = (V \setminus S, \sigma', \mu')$, where σ' and μ' are the restrictions of σ and μ to the remaining vertices, is a fuzzy quasi outerplanar graph.

The properties of a fuzzy k-quasi apex outerplanar graph include:

- If $k = 0$, the fuzzy k-quasi apex outerplanar graph itself is already a fuzzy quasi outerplanar graph.
- When $k = 1$, the fuzzy graph corresponds to a fuzzy quasi apex outerplanar graph where the removal of a single vertex makes the graph fuzzy quasi outerplanar.

Theorem 75. *A fuzzy k-quasi apex outerplanar graph can be transformed into a fuzzy outer-quasi-planar graph.*

Proof. Obviously holds. □

2.3.8 Fuzzy almost planar graph

The definitions of an almost planar graph and an almost outerplanar graph are extended to the context of fuzzy graphs as follows.

Definition 76. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy almost planar graph* if there exists an edge $e \in E(\psi)$ such that the fuzzy graph $\psi - e$ (i.e., the fuzzy graph obtained by removing the edge e from ψ) is a fuzzy planar graph.

Definition 77. A fuzzy graph $\psi = (V, \sigma, \mu)$ is called a *fuzzy almost outerplanar graph* if there exists an edge $e \in E(\psi)$ such that the fuzzy graph $\psi - e$ (i.e., the fuzzy graph obtained by removing the edge e from ψ) is a fuzzy outerplanar graph.

Theorem 78. *An fuzzy almost planar graph can be transformed into a almost planar graph.*

Proof. Obviously holds. □

Theorem 79. *An fuzzy almost outerplanar graph can be transformed into a almost outerplanar graph.*

Proof. Obviously holds. □

2.4 Intuitionistic fuzzy planar graphs

Intuitionistic fuzzy graphs are an extended version of fuzzy graphs and have been the subject of extensive study for over 15 years [247, 328, 432, 447]. Intuitionistic fuzzy planar graphs [49] are, in turn, an extension of fuzzy planar graphs. The concept of intuitionistic fuzzy sets [70–72, 176, 211, 358, 517], which is well-known in set theory, is closely related to intuitionistic fuzzy graphs. The definitions of intuitionistic fuzzy graphs and intuitionistic fuzzy planar graphs are provided below.

Definition 80 (Intuitionistic Fuzzy Graph (IFG))**.** [432] Let $G = (V, E)$ be a classical graph where V denotes the set of vertices and E denotes the set of edges. An *Intuitionistic Fuzzy Graph* (IFG) on G, denoted $G_{IF} = (A, B)$, is defined as follows:

1. (μ_A, ν_A) is an *Intuitionistic Fuzzy Set (IFS)* on the vertex set V. For each vertex $x \in V$, the degree of membership $\mu_A(x) \in [0, 1]$ and the degree of non-membership $\nu_A(x) \in [0, 1]$ satisfy:

$$\mu_A(x) + \nu_A(x) \leq 1$$

The value $1 - \mu_A(x) - \nu_A(x)$ represents the hesitancy or uncertainty regarding the membership of x in the set.

2. (μ_B, ν_B) is an *Intuitionistic Fuzzy Relation (IFR)* on the edge set E. For each edge $(x, y) \in E$, the degree of membership $\mu_B(x, y) \in [0, 1]$ and the degree of non-membership $\nu_B(x, y) \in [0, 1]$ satisfy:

$$\mu_B(x, y) + \nu_B(x, y) \leq 1$$

Additionally, the following constraints must hold for all $x, y \in V$:

$$\mu_B(x, y) \leq \mu_A(x) \wedge \mu_A(y)$$

$$\nu_B(x, y) \leq \nu_A(x) \vee \nu_A(y)$$

In this definition:

- $\mu_A(x)$ and $\nu_A(x)$ represent the degree of membership and non-membership of the vertex x, respectively.
- $\mu_B(x, y)$ and $\nu_B(x, y)$ represent the degree of membership and non-membership of the edge (x, y), respectively.
- If $\nu_A(x) = 0$ and $\nu_B(x, y) = 0$ for all $x \in V$ and $(x, y) \in E$, then the Intuitionistic Fuzzy Graph reduces to a Fuzzy Graph.

Theorem 81. *[235] The following are examples of related graph classes, including but not limited to:*

- *Strong Intuitionistic Fuzzy Graphs [23]*
- *Perfect intuitionistic fuzzy graphs [247]*
- *Intuitionistic fuzzy competition graphs[467]*
- *Intuitionistic fuzzy threshold graphs[554]*
- *Balanced Intuitionistic Fuzzy Graphs[324]*
- *Bipolar Intuitionistic Fuzzy Competition Graphs[185]*
- *Intuitionistic Felicitous Fuzzy Graphs [95]*
- *Intuitionistic Fuzzy Soft Expert Graphs [533]*
- *Intuitionistic fuzzy tolerance graphs[468]*
- *Intuitionistic fuzzy planar graphs[50]*
- *Edge regular intuitionistic fuzzy graph[325]*
- *Edge Irregular Intuitionistic Fuzzy Graphs[410]*
- *m-Neighbourly Irregular Instuitionistic Fuzzy Graphs[376]*
- *R-edge regular intuitionistic fuzzy graphs[43]*
- *Perfectly Edge-Regular Intuitionistic Fuzzy Graphs[2]*
- *Edge Regular Intuitionistic Fuzzy M-Polargraphs[252]*
- *Intuitionistic fuzzy labeling graphs[464]*
- *Regular Interval-Valued Intuitionistic Fuzzy Graphs[1]*
- *Anti intuitionistic fuzzy graph[307]*
- *Interval-valued intuitionistic fuzzy graphs[346]*
- *Intuitionistic fuzzy directed hypergraphs[408]*
- *Complex intuitionistic fuzzy graphs[62]*
- *Complex t-Intuitionistic Fuzzy Graph[329]*
- *Intuitionistic fuzzy incidence graphs[414]*
- *Intuitionistic fuzzy k-partite hypergraphs[405, 406]*
- *Irregular Intuitionistic Fuzzy Graphs[391]*
- *intuitionistic L-fuzzy graph[502, 514]*
- *Bipolar intuitionistic anti fuzzy graphs[184]*
- *Intuitionistic anti-fuzzy graphs[404]*
- *intuitionistic product fuzzy graphs[518]*
- *intuitionistic k-partitioned fuzzy graph[435]*
- *intutionistic fuzzy multigraphs[73]*

Proof. Refer to each reference as needed. □

2.4.1 Intuitionistic fuzzy planar graph

The Intuitionistic fuzzy planar graph is an extension of the fuzzy planar graph. These graphs have been studied in a similar manner to fuzzy planar graphs. The definition of the Intuitionistic fuzzy planar graph is provided below[49, 466].

Definition 82. [49] Let G be an intuitionistic fuzzy multigraph[103, 444, 483], and let $P_1, P_2, \ldots, P_z$ be the points of intersection between the edges for a given geometrical representation. The graph G is said to be an *intuitionistic fuzzy planar graph* with an intuitionistic fuzzy planarity value $f = (f_M, f_N)$, where

$$f = (f_M, f_N) = \left(\frac{1}{1 + (M_{P_1} + M_{P_2} + \cdots + M_{P_z})}, \frac{1}{1 + (N_{P_1} + N_{P_2} + \cdots + N_{P_z})} \right).$$

Here, M_{P_i} and N_{P_i} represent the membership and non-membership degrees at each point of intersection P_i, respectively.

The intuitionistic fuzzy planarity value $f = (f_M, f_N)$ is bounded such that $0 < f_M \leq 1$ and $0 < f_N \leq 1$. If there are no points of intersection for a certain geometrical representation of the intuitionistic fuzzy planar graph, then its intuitionistic fuzzy planarity value is $(1, 1)$.

Proposition 83. *An intuitionistic fuzzy planar graph is an intuitionistic fuzzy graph.*

Proof. Obviously holds. □

Theorem 84. *An intuitionistic fuzzy planar graph can be transformed into a fuzzy planar graph.*

Proof. Obviously holds. □

2.4.2 Intuitionistic fuzzy outerplanar graph

We define the intuitionistic fuzzy outerplanar graph, which is an extension of the fuzzy outerplanar graph.

Definition 85. An *intuitionistic fuzzy outerplanar graph* $\psi = (V, \mu, \nu)$ is an intuitionistic fuzzy graph where:

- It can be embedded in a plane such that every vertex $v \in V$ lies on the boundary of the exterior region of the embedding.
- Similar to intuitionistic fuzzy graphs, $\mu : V \to [0, 1]$ and $\nu : V \to [0, 1]$ represent the membership and non-membership degrees of each vertex, respectively.
- For an intuitionistic fuzzy graph to be outerplanar, there must be no edge intersection while ensuring all vertices are positioned on the outer face.

If $i(\psi)$ represents the number of vertices not lying on the boundary of the outer face, then an intuitionistic fuzzy outerplanar graph satisfies $i(\psi) = 0$. In contrast, if $i(\psi) \neq 0$, the graph is considered a non-outerplanar intuitionistic fuzzy graph.

These definitions extend the classical notions of outerplanarity to the intuitionistic fuzzy domain, allowing for the representation and analysis of uncertainty in graph structures.

Theorem 86. *An intuitionistic fuzzy outerplanar graph can be transformed into a fuzzy outerplanar graph.*

Proof. Obviously holds. □

Theorem 87. *An intuitionistic fuzzy outerplanar graph is an intuitionistic fuzzy planar graph.*

Proof. Obviously holds. □

2.4.3 Intuitionistic fuzzy apex graph

We define the intuitionistic fuzzy apex graph, which is an extension of the fuzzy apex graph.

Definition 88. An *intuitionistic fuzzy apex graph* $\psi = (V, \mu, \nu)$ is an intuitionistic fuzzy graph that can be transformed into an intuitionistic fuzzy planar graph by removing a single vertex $v \in V$. The removed vertex is called an *apex* of the intuitionistic fuzzy graph ψ.

Formally, ψ is an intuitionistic fuzzy apex graph if there exists a vertex $v \in V$ such that the subgraph $\psi' = (V \setminus \{v\}, \mu', \nu')$, where μ' and ν' are the restrictions of the membership and non-membership functions μ and ν on the remaining vertices, is an intuitionistic fuzzy planar graph.

The intuitionistic fuzzy apex graph retains the following properties:

- $\mu : V \to [0, 1]$ assigns a membership degree to each vertex in V.
- $\nu : V \to [0, 1]$ assigns a non-membership degree to each vertex in V.
- If the intuitionistic fuzzy apex graph ψ itself is an intuitionistic fuzzy planar graph, then every vertex in V can be considered an apex.

Similar to classical apex graphs, an intuitionistic fuzzy apex graph may have multiple apex vertices. Additionally, if ψ is a null intuitionistic fuzzy graph (i.e., $V = \emptyset$), it is trivially an intuitionistic fuzzy apex graph.

Theorem 89. *An intuitionistic fuzzy apex graph can be transformed into a fuzzy apex graph.*

Proof. Obviously holds. □

Theorem 90. *An intuitionistic fuzzy apex graph can be transformed into an intuitionistic fuzzy planar graph.*

Proof. Obviously holds. □

2.4.4 Intuitionistic Fuzzy Quasi-Planar Graph

The definition of an Intuitionistic Fuzzy Quasi-Planar Graph is provided below. This graph is an extension of the Fuzzy Quasi-Planar Graph.

Definition 91. An *Intuitionistic Fuzzy Quasi-Planar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that no three intuitionistic fuzzy edges are pairwise crossing.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy Quasi-Planar Graph* if there exists a drawing D in which any set of three intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 92. An *Intuitionistic Fuzzy k-Quasi-Planar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that no k intuitionistic fuzzy edges are pairwise crossing.

Formally, let $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ be an intuitionistic fuzzy graph where:

- $\mu_A : V \to [0, 1]$ assigns a membership degree to each vertex $v \in V$, and $\nu_A : V \to [0, 1]$ assigns a non-membership degree such that $\mu_A(v) + \nu_A(v) \leq 1$.
- $\mu_B : V \times V \to [0, 1]$ assigns a membership degree to each edge (u, v), and $\nu_B : V \times V \to [0, 1]$ assigns a non-membership degree such that $\mu_B(u, v) + \nu_B(u, v) \leq 1$ and $\mu_B(u, v) \leq \min\{\mu_A(u), \mu_A(v)\}$.

The intuitionistic fuzzy graph ψ_{IF} is called *Intuitionistic Fuzzy k-Quasi-Planar* if there exists a drawing D in which any set of k intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 93. *An Intuitionistic Fuzzy k-Quasi-Planar Graph can be transformed into a Intuitionistic Fuzzy Quasi-Planar graph.*

Proof. Obviously holds. □

Theorem 94. *An Intuitionistic Fuzzy k-Quasi-Planar Graph can be transformed into a Fuzzy k-Quasi-Planar graph.*

Proof. Obviously holds. □

2.4.5 Intuitionistic Fuzzy Quasi-Outerplanar Graph

The definition of an Intuitionistic Fuzzy Quasi-Outerplanar Graph is provided below. This graph is an extension of the Fuzzy Quasi-Outerplanar Graph.

Definition 95. An *Intuitionistic Fuzzy Quasi-Outerplanar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three intuitionistic fuzzy edges are pairwise crossing.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy Quasi-Outerplanar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of three intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Definition 96. An *Intuitionistic Fuzzy Outer-k-Quasi-Planar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k intuitionistic fuzzy edges are pairwise crossing.

Formally, let $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ be an intuitionistic fuzzy graph where:

- μ_A, ν_A, μ_B, and ν_B are as defined above.

The intuitionistic fuzzy graph ψ_{IF} is called *Intuitionistic Fuzzy Outer-k-Quasi-Planar* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k intuitionistic fuzzy edges do not all intersect each other at distinct points in their interiors.

Theorem 97. *An Intuitionistic Fuzzy Outer-k-Quasi-Planar Graph can be transformed into a Fuzzy Outer-k-Quasi-Planar graph.*

Proof. Obviously holds. □

Theorem 98. *An Intuitionistic Fuzzy Quasi-Outerplanar Graph can be transformed into a Fuzzy Quasi-Outerplanar Graph.*

Proof. Obviously holds. □

2.4.6 Intuitionistic Fuzzy Quasi Apex Graph

The definition of an Intuitionistic Fuzzy Quasi Apex Graph is provided below. This graph is an extension of the Fuzzy Quasi Apex Graph.

Definition 99. An *Intuitionistic Fuzzy k-Quasi Apex Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be transformed into an intuitionistic fuzzy quasi planar graph by removing at most k vertices.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy k-Quasi Apex Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced intuitionistic fuzzy subgraph $\psi'_{IF} = (V \setminus S, \mu'_A, \nu'_A, \mu'_B, \nu'_B)$, where $\mu'_A, \nu'_A, \mu'_B, \nu'_B$ are the restrictions of $\mu_A, \nu_A, \mu_B, \nu_B$ on the remaining vertices, is an intuitionistic fuzzy Quasi planar graph.

Theorem 100. *An Intuitionistic Fuzzy k-Quasi Apex Graph can be transformed into a Fuzzy k-Quasi Apex Graph.*

Proof. Obviously holds. □

2.4.7 Intuitionistic Fuzzy Quasi Apex Outerplanar Graph

The definition of an Intuitionistic Fuzzy Quasi Apex Outerplanar Graph is provided below. This graph is an extension of the Fuzzy Quasi Apex Outerplanar Graph.

Definition 101. An *Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph* $\psi_{IF} = (V, \mu_A, \nu_A, \mu_B, \nu_B)$ is an intuitionistic fuzzy graph that can be transformed into an intuitionistic fuzzy quasi outerplanar graph by removing at most k vertices.

Formally, ψ_{IF} is called an *Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced intuitionistic fuzzy subgraph $\psi'_{IF} = (V \setminus S, \mu'_A, \nu'_A, \mu'_B, \nu'_B)$, where $\mu'_A, \nu'_A, \mu'_B, \nu'_B$ are the restrictions of $\mu_A, \nu_A, \mu_B, \nu_B$ on the remaining vertices, is an intuitionistic fuzzy Quasi outerplanar graph.

Theorem 102. *An Intuitionistic Fuzzy k-Quasi Apex Outerplanar Graph can be transformed into a Fuzzy k-Quasi Apex Outerplanar Graph.*

Proof. Obviously holds. □

2.4.8 Intuitionistic fuzzy almost planar graph

The definitions of an Intuitionistic fuzzy almost planar graph and an almost outerplanar graph are extended to the context of Intuitionistic fuzzy graphs as follows.

Definition 103. An *Intuitionistic fuzzy graph* $\psi_{IF} = (V, (\mu_A, \nu_A), (\mu_B, \nu_B))$ is called an *Intuitionistic fuzzy almost planar graph* if there exists an edge $e \in E(\psi_{IF})$ such that the Intuitionistic fuzzy graph $\psi_{IF} - e$ (i.e., the Intuitionistic fuzzy graph obtained by removing the edge e from ψ_{IF}) is an Intuitionistic fuzzy planar graph.

Definition 104. An *Intuitionistic fuzzy graph* $\psi_{IF} = (V, (\mu_A, \nu_A), (\mu_B, \nu_B))$ is called an *Intuitionistic fuzzy almost outerplanar graph* if there exists an edge $e \in E(\psi_{IF})$ such that the Intuitionistic fuzzy graph $\psi_{IF} - e$ (i.e., the Intuitionistic fuzzy graph obtained by removing the edge e from ψ_{IF}) is an Intuitionistic fuzzy outerplanar graph.

Theorem 105. *An intuitionistic fuzzy almost planar graph can be transformed into a fuzzy almost planar graph.*

Proof. Obviously holds. □

Theorem 106. *An intuitionistic fuzzy almost outerplanar graph can be transformed into a fuzzy almost outerplanar graph.*

Proof. Obviously holds. □

2.5 Neutrosophic graph and planar graph

First, the definition of a neutrosophic graph is provided. Similar to fuzzy graphs, neutrosophic graphs have been the focus of extensive research [28, 33, 130, 237, 238, 276, 297, 319, 463, 493, 497]. Additionally, the concept of neutrosophic sets[128, 212, 378, 540] in set theory is closely related to neutrosophic graphs. The definition is given below[237, 497].

Definition 107. [237, 497] A neutrosophic graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is defined as a graph where $\sigma_i : V \to [0, 1]$, $\mu_i : E \to [0, 1]$, and for every $v_i v_j \in E$, the following condition holds: $\mu(v_i v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$.

1. σ is called the neutrosophic vertex set.
2. μ is called the neutrosophic edge set.
3. $|V|$ is called the order of NTG, and it is denoted by $O(NTG)$.
4. $\sum_{v \in V} \sigma(v)$ is called the neutrosophic order of NTG, and it is denoted by $On(NTG)$.
5. $|E|$ is called the size of NTG, and it is denoted by $S(NTG)$.
6. $\sum_{e \in E} \mu(e)$ is called the neutrosophic size of NTG, and it is denoted by $Sn(NTG)$.

Theorem 108. *[235] The following are examples of related Neutrosophic graph classes, including but not limited to:*

- *Bipolar Neutrosophic Graphs [31]*
- *Interval-Valued Neutrosophic Graphs [486]*
- *Single-Valued Neutrosophic Graphs [35, 131]*
- *Interval Complex Neutrosophic Graphs [129, 300]*
- *Neutrosophic Hypergraphs [250]*
- *Neutrosophic Vague Line Graphs [300]*
- *Neutrosophic Vague Graphs [301]*
- *Single-Valued Neutrosophic Signed Graphs [386]*
- *Neutrosophic Soft Rough Graphs [28]*
- *Neutrosophic Incidence Graphs [32]*
- *Fermatean Neutrosophic Dombi Fuzzy Graphs [474]*
- *Interval valued pentapartitioned neutrosophic graphs [129]*
- *Single valued pentapartitioned neutrosophic graphs[174]*

- *pentapartitioned neutrosophic graphs[443]*
- *t-Neutrosophic Fuzzy graph[327]*
- *Complex t-Neutrosophic Graph[330]*
- *Regular neutrosophic graphs[297]*
- *Q-neutrosophic soft graphs[532]*
- *Balanced Neutrosophic Graphs[490]*
- *Neutrosophic minimum spanning tree graph[363]*

Proof. Refer to each reference as needed. □

2.5.1 Neutrosophic planar graph

A neutrosophic planar graph is an extension of fuzzy planar graphs and intuitionistic fuzzy planar graphs [37, 146, 375].

Definition 109. [320] Let $G = (A, B)$ be a *neutrosophic multigraph*, where B contains edges $(ab, T_B(ab), I_B(ab), F_B(ab))$ and $(cd, T_B(cd), I_B(cd), F_B(cd))$ that intersect at a point P. The intersecting value at point P is defined as

$$S_P = ((S_T)_P, (S_I)_P, (S_F)_P) = \left(\frac{(S_T)_{ab} + (S_T)_{cd}}{2}, \frac{(S_I)_{ab} + (S_I)_{cd}}{2}, \frac{(S_F)_{ab} + (S_F)_{cd}}{2} \right).$$

As the number of intersection points in a neutrosophic multigraph increases, its planarity decreases. Thus, for a neutrosophic multigraph, S_P is inversely proportional to the planarity.

Definition 110. [37,146,375] A neutrosophic multigraph G with intersection points $P_1, P_2, \ldots, P_z$ between the edges in a certain geometrical representation is called a *neutrosophic planar graph* if the neutrosophic planarity value $f = (f_T, f_I, f_F)$ satisfies: $f_T = f_I = f_F = 1$.

The neutrosophic planarity value is computed as:

$$f = (f_T, f_I, f_F) = \left(\frac{1}{1 + \{(S_T)_{P_1} + (S_T)_{P_2} + \ldots + (S_T)_{P_z}\}}, \frac{1}{1 + \{(S_I)_{P_1} + (S_I)_{P_2} + \ldots + (S_I)_{P_z}\}}, \frac{1}{1 + \{(S_F)_{P_1} + (S_F)_{P_2} + \ldots + (S_F)_{P_z}\}} \right),$$

where S_T, S_I, and S_F represent the truth, indeterminacy, and falsity values of the intersections at each point P_i. For the graph to be considered neutrosophically planar, we must have:

$$(S_T)_{P_1} + (S_T)_{P_2} + \ldots + (S_T)_{P_z} = 0, \quad (S_I)_{P_1} + (S_I)_{P_2} + \ldots + (S_I)_{P_z} = 0, \quad (S_F)_{P_1} + (S_F)_{P_2} + \ldots + (S_F)_{P_z} = 0.$$

Thus, the neutrosophic planarity value must be $(f_T, f_I, f_F) = (1, 1, 1)$, indicating no geometrical intersections and perfect planarity in all neutrosophic dimensions.

Theorem 111. *Let $G_N = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic planar graph, where:*

- *$\sigma : V \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.*
- *$\mu : E \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.*

By setting the indeterminacy $\sigma_I(v) = 0$ and the falsity $\sigma_F(v) = 0$ for all vertices $v \in V$, and similarly setting $\mu_I(e) = 0$ and $\mu_F(e) = 0$ for all edges $e \in E$, the graph G_N can be transformed into a fuzzy planar graph G_F. Specifically:

- *The membership degree of the fuzzy graph G_F is $\sigma(v) = \sigma_T(v)$ for each vertex v.*
- *The fuzzy relation $\mu(u, v) = \mu_T(u, v)$ for each edge (u, v).*

Proof. 1. *Vertex Transformation:* For each vertex $v \in V$, the representation is $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ By setting $\sigma_I(v) = 0$ and $\sigma_F(v) = 0$, we obtain $\sigma(v) = (\sigma_T(v), 0, 0)$. Therefore, in the fuzzy graph representation, $\sigma(v) = \sigma_T(v)$, which corresponds to the membership degree of the vertex v.

2. *Edge Transformation:* For each edge $e \in E$, the neutrosophic representation is $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$. By setting $\mu_I(e) = 0$ and $\mu_F(e) = 0$, we obtain $\mu(e) = (\mu_T(e), 0, 0)$. Thus, the membership degree of the edge $\mu(u, v) = \mu_T(u, v)$ corresponds to the fuzzy relation in the resulting fuzzy graph.

3. *Planarity:* Since G_N is a planar graph and setting $\sigma_I(v)$, $\sigma_F(v)$, $\mu_I(e)$, and $\mu_F(e)$ to zero does not alter the graph's structure or the absence of intersecting edges, the resulting graph G_F retains the planarity properties.

Hence, the neutrosophic planar graph G_N is transformed into a fuzzy planar graph G_F by setting the indeterminacy and falsity values to zero. □

Theorem 112. *Let $G_N = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic planar graph. By setting the indeterminacy degree $\sigma_I(v) = 0$ for all vertices $v \in V$ and $\mu_I(e) = 0$ for all edges $e \in E$, the graph G_N can be transformed into an intuitionistic fuzzy planar graph G_{IF}. Specifically:*

- *The membership degree $\mu_A(v) = \sigma_T(v)$ for each vertex v.*
- *The non-membership degree $\nu_A(v) = \sigma_F(v)$ for each vertex v.*

Similarly, for each edge e, the membership and non-membership degrees are $\mu_{IF}(e) = \mu_T(e)$ and $\nu_{IF}(e) = \mu_F(e)$, respectively. The resulting graph G_{IF} satisfies the properties of an intuitionistic fuzzy planar graph.

Proof. 1. *Vertex Transformation:* For each vertex $v \in V$, the representation is $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$. By setting $\sigma_I(v) = 0$, we obtain $\sigma(v) = (\sigma_T(v), 0, \sigma_F(v))$. Defining $\mu_A(v) = \sigma_T(v)$ and $\nu_A(v) = \sigma_F(v)$, we ensure $\mu_A(v) + \nu_A(v) = \sigma_T(v) + \sigma_F(v) \leq 1$.

2. *Edge Transformation:* For each edge $e \in E$, the neutrosophic representation is $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$. By setting $\mu_I(e) = 0$, we obtain $\mu(e) = (\mu_T(e), 0, \mu_F(e))$. Defining $\mu_{IF}(e) = \mu_T(e)$ and $\nu_{IF}(e) = \mu_F(e)$, we have $\mu_{IF}(e) + \nu_{IF}(e) \leq 1$.

3. *Planarity:* Since G_N is a planar graph, there exists an embedding in the plane such that no edges intersect except at vertices. Thus, the resulting graph G_{IF} is also planar, with fuzzy representation characteristics maintained.

Therefore, by setting the indeterminacy values $\sigma_I(v) = 0$ and $\mu_I(e) = 0$, the neutrosophic planar graph G_N transforms into an intuitionistic fuzzy planar graph G_{IF}. □

If there are no points of intersection for a given geometrical representation of the neutrosophic planar graph, its neutrosophic planarity value is $(1, 1, 1)$. In such a case, the underlying crisp graph of this neutrosophic graph corresponds to a classical planar graph. As f_T and f_I decrease and f_F increases, the number of intersection points between edges increases or decreases, respectively, and the degree of planarity decreases or increases accordingly.

2.5.2 Neutrosophic outerplanar graph

We introduce a new definition for the Neutrosophic outerplanar graph. The Neutrosophic outerplanar graph extends the concepts of Fuzzy outerplanar Graph and Intuitionistic Fuzzy outerplanar graph as described below.

Definition 113 (Neutrosophic Outerplanar Graph)**.** A *neutrosophic outerplanar graph*

$$G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$$

is a neutrosophic multigraph such that:

- $\sigma : A \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in A$.
- $\mu : B \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in B$.

The graph G is called neutrosophic outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of the exterior region, satisfying the following conditions:

1. No two edges intersect except at their endpoints.
2. All vertices are positioned on the outer face of the graph embedding.

If there are intersection points $P_1, P_2, \ldots, P_z$ in the embedding, the neutrosophic outerplanarity value $f = (f_T, f_I, f_F)$ is defined as

$$f = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_I)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}} \right),$$

where $S_{P_i} = ((S_T)_{P_i}, (S_I)_{P_i}, (S_F)_{P_i})$ denotes the intersecting values at point P_i. If no intersections exist, $f = (1, 1, 1)$.

Next, we will provide the definitions of a maximal neutrosophic outerplanar graph and a minimally neutrosophic non-outerplanar graph.

Definition 114. If an edge cannot be added to a neutrosophic outerplanar graph ψ without sacrificing its outerplanarity property, then the graph is called a *maximal neutrosophic outerplanar graph*. The graphs shown in Figures are examples of maximal neutrosophic outerplanar graphs.

Definition 115. A neutrosophic planar graph ψ is considered *minimally neutrosophic non-outerplanar* if $i(\psi) \neq 0$ with at most one vertex $v \in V$ such that $\sigma_I(v) > 0$ or $\sigma_F(v) > 0$ in the interior region.

Theorem 116. *A neutrosophic outerplanar graph can be transformed into a fuzzy outerplanar graph by setting the indeterminacy and falsity degrees $\sigma_I(v)$ and $\sigma_F(v)$ for all vertices, and $\mu_I(e)$ and $\mu_F(e)$ for all edges, to zero.*

Proof. Let $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic outerplanar graph, where:

- $\sigma : A \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ to each vertex $v \in A$.
- $\mu : B \to [0, 1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ to each edge $e \in B$.

Define a new fuzzy outerplanar graph $\psi = (V, \sigma', \mu')$ where:

- $V = A$ (the set of vertices remains the same),
- $\sigma' : V \to [0, 1]$ is defined by $\sigma'(v) = \sigma_T(v)$,
- $\mu' : V \times V \to [0, 1]$ is defined by $\mu'(u, v) = \mu_T(e)$ where e is the edge between u and v.

By setting $\sigma_I(v) = 0$ and $\sigma_F(v) = 0$ for all $v \in V$, the original triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ reduces to a single membership value $\sigma_T(v)$.

Similarly, for each edge e, setting $\mu_I(e) = 0$ and $\mu_F(e) = 0$ reduces $(\mu_T(e), \mu_I(e), \mu_F(e))$ to the single value $\mu_T(e)$.

This transformation results in:

$$\mu'(u, v) \leq \min\{\sigma'(u), \sigma'(v)\},$$

satisfying the conditions of a fuzzy outerplanar graph.

Additionally, since all vertices lie on the boundary of the exterior region in the neutrosophic embedding, the transformed graph ψ maintains the outerplanar property. Therefore, the resulting graph $\psi = (V, \sigma', \mu')$ is a valid fuzzy outerplanar graph. □

2.5.3 Neutrosophic apex graph

The definition of a neutrosophic apex graph is provided below. This graph extends the concepts of fuzzy apex graphs and intuitionistic fuzzy apex graphs, as detailed in the following sections.

Definition 117. A *neutrosophic apex graph* $\psi = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a neutrosophic graph that can be transformed into a neutrosophic planar graph by removing a single vertex $v \in V$. The removed vertex is called an *apex* of the neutrosophic graph ψ.

Formally, ψ is a neutrosophic apex graph if there exists a vertex $v \in V$ such that the subgraph $\psi' = (V \setminus \{v\}, E', \sigma', \mu')$, where E' is the set of edges remaining after the removal of v, and σ' and μ' are the restrictions of the membership functions σ and μ on the remaining vertices and edges, is a neutrosophic planar graph.

The neutrosophic apex graph retains the following properties:

- $\sigma : V \to [0, 1]^3$ assigns a triple $(\sigma_T(v), \sigma_I(v), \sigma_F(v))$ representing the truth, indeterminacy, and falsity membership degrees to each vertex $v \in V$.

- $\mu : E \to [0,1]^3$ assigns a triple $(\mu_T(e), \mu_I(e), \mu_F(e))$ representing the truth, indeterminacy, and falsity membership degrees to each edge $e \in E$.
- If the neutrosophic apex graph ψ itself is neutrosophic planar, then every vertex in V can be considered an apex.

Similar to classical apex graphs, a neutrosophic apex graph may have multiple apex vertices. Additionally, if ψ is the null neutrosophic graph (i.e., $V = \emptyset$), it is trivially a neutrosophic apex graph.

Definition 118. A *neutrosophic k-apex graph* $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a neutrosophic graph that can be transformed into a neutrosophic planar graph by removing at most k vertices from A.

If $k = 0$, the neutrosophic k-apex graph itself is already a neutrosophic planar graph. When $k = 1$, the graph corresponds to a neutrosophic apex graph as defined earlier.

2.5.4 Neutrosophic Apex Outerplanar Graph

The definition of a neutrosophic Apex Outerplanar Graph is provided below. This graph extends the concepts of fuzzy Apex Outerplanar Graph and intuitionistic fuzzy Apex Outerplanar Graph, as detailed in the following sections.

Definition 119. A *Neutrosophic Apex Outerplanar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be transformed into a neutrosophic outerplanar graph by removing a single vertex $v \in V$.

Formally, ψ_N is called a *Neutrosophic Apex Outerplanar Graph* if there exists a vertex $v \in V$ such that the induced neutrosophic subgraph $\psi'_N = (V \setminus \{v\}, T'_A, I'_A, F'_A, T'_B, I'_B, F'_B)$, where $T'_A, I'_A, F'_A, T'_B, I'_B, F'_B$ are the restrictions of $T_A, I_A, F_A, T_B, I_B, F_B$ to the remaining vertices, is a neutrosophic outerplanar graph.

2.5.5 Neutrosophic Quasi-Planar Graph

The definition of a Neutrosophic Quasi-Planar Graph is provided below. This graph extends the concepts of fuzzy Quasi-Planar Graph and intuitionistic fuzzy Quasi-Planar Graph, as detailed in the following sections.

Definition 120. A *Neutrosophic Quasi-Planar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that no three neutrosophic edges are pairwise crossing.

Formally, ψ_N is called a *Neutrosophic Quasi-Planar Graph* if there exists a drawing D in which any set of three neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 121. A *Neutrosophic Outer-k-Quasi-Planar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k neutrosophic edges are pairwise crossing.

Formally, the neutrosophic graph ψ_N is called a *Neutrosophic Outer-k-Quasi-Planar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k neutrosophic edges do not all intersect each other at distinct points in their interiors.

2.5.6 Neutrosophic Quasi-Outerplanar Graph

The definition of a Neutrosophic Quasi-Outerplanar Graph is provided below. This graph extends the concepts of fuzzy Quasi-Outerplanar Graph and intuitionistic fuzzy Quasi-Outerplanar Graph, as detailed in the following sections.

Definition 122. A *Neutrosophic Quasi-Outerplanar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three neutrosophic edges are pairwise crossing.

Formally, ψ_N is called a *Neutrosophic Quasi-Outerplanar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of three neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 123. A *Neutrosophic k-Quasi-Planar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be drawn in the plane such that no k neutrosophic edges are pairwise crossing.

Formally, let $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ be a neutrosophic graph where:

- $T_A : V \to [0,1]$, $I_A : V \to [0,1]$, and $F_A : V \to [0,1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees of each vertex, respectively, satisfying $0 \leq T_A(v) + I_A(v) + F_A(v) \leq 1$.
- $T_B : V \times V \to [0,1]$, $I_B : V \times V \to [0,1]$, and $F_B : V \times V \to [0,1]$ represent the truth-membership, indeterminacy-membership, and falsity-membership degrees for each edge (u, v), satisfying $T_B(u,v) + I_B(u,v) + F_B(u,v) \leq 1$ and $T_B(u,v) \leq \min\{T_A(u), T_A(v)\}$.

2.5.7 Neutrosophic Quasi Apex Graph

The definition of a Neutrosophic Quasi Apex Graphis provided below. This graph extends the concepts of fuzzy Quasi Apex Graph and intuitionistic fuzzy Quasi Apex Graph, as detailed in the following sections.

Definition 124. A *Neutrosophic k-Quasi Apex Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be transformed into a neutrosophic planar graph by removing at most k vertices.

Formally, ψ_N is called a *Neutrosophic k-Quasi Apex Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced neutrosophic subgraph $\psi'_N = (V \setminus S, T'_A, I'_A, F'_A, T'_B, I'_B, F'_B)$, where $T'_A, I'_A, F'_A, T'_B, I'_B, F'_B$ are the restrictions of $T_A, I_A, F_A, T_B, I_B, F_B$ to the remaining vertices, is a neutrosophic quasi-planar graph.

2.5.8 Neutrosophic Quasi Apex Outerplanar Graph

The definition of a Neutrosophic Quasi Apex Outerplanar Graph is provided below. This graph extends the concepts of fuzzy Quasi Apex Outerplanar Graph and intuitionistic fuzzy Quasi Apex Outerplanar Graph, as detailed in the following sections.

Definition 125. A *Neutrosophic k-Quasi Apex Outerplanar Graph* $\psi_N = (V, T_A, I_A, F_A, T_B, I_B, F_B)$ is a neutrosophic graph that can be transformed into a neutrosophic outerplanar graph by removing at most k vertices.

Formally, ψ_N is called a *Neutrosophic k-Quasi Apex Outerplanar Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced neutrosophic subgraph $\psi'_N = (V \setminus S, T'_A, I'_A, F'_A, T'_B, I'_B, F'_B)$, where $T'_A, I'_A, F'_A, T'_B, I'_B, F'_B$ are the restrictions of $T_A, I_A, F_A, T_B, I_B, F_B$ to the remaining vertices, is a neutrosophic quasi outerplanar graph.

2.5.9 Neutrosophic almost planar graph

The definitions of a neutrosophic almost planar graph and an neutrosophic almost outerplanar graph are extended to the context of neutrosophic fuzzy graphs as follows.

Definition 126. A *neutrosophic graph* $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is called a *neutrosophic almost planar graph* if there exists an edge $e \in E(NTG)$ such that the neutrosophic graph $NTG - e$ (i.e., the neutrosophic graph obtained by removing the edge e from NTG) is a neutrosophic planar graph.

Definition 127. A *neutrosophic graph* $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$ is called a *neutrosophic almost outerplanar graph* if there exists an edge $e \in E(NTG)$ such that the neutrosophic graph $NTG - e$ (i.e., the neutrosophic graph obtained by removing the edge e from NTG) is a neutrosophic outerplanar graph.

Theorem 128. *An neutrosophic almost planar graph can be transformed into an intuitionistic fuzzy almost planar graph.*

Proof. Obviously holds. □

Theorem 129. *An neutrosophic almost outerplanar graph can be transformed into an intuitionistic fuzzy almost outerplanar graph.*

Proof. Obviously holds. □

2.6 Turiyam Neutrosophic Graph

Research on Turiyam Neutrosophic Graphs, which incorporate parameters into Neutrosophic Graphs, is currently being conducted [244–246]. These graphs are a graphical representation of the Turiyam Neutrosophic Set [246, 487]. Similar concepts include four-valued logic [90, 150]. The definition is provided below.

Definition 130 (Turiyam Neutrosophic Graph)**.** [244–246] Let $G = (V, E)$ be a classical graph with a finite set of vertices $V = \{v_i : i = 1, 2, \ldots, n\}$ and edges $E = \{(v_i, v_j) : i, j = 1, 2, \ldots, n\}$. A *Turiyam Neutrosophic Graph* of G, denoted $G^T = (V^T, E^T)$, is defined as follows:

1. *Turiyam Neutrosophic Vertex Set*: For each vertex $v_i \in V$, the Turiyam Neutrosophic graph assigns the following mappings:
$$t(v_i), iv(v_i), fv(v_i), lv(v_i) : V \to [0, 1],$$
where:
   - $t(v_i)$ is the truth value (tv) of the vertex v_i,
   - $iv(v_i)$ is the indeterminacy value (iv) of v_i,
   - $fv(v_i)$ is the falsity value (fv) of v_i,

- $lv(v_i)$ is the Turiyam Neutrosophic state (or liberal value) (lv) of v_i,

for all $v_i \in V$, such that the following condition holds for each vertex:

$$0 \leq t(v_i) + iv(v_i) + fv(v_i) + lv(v_i) \leq 4.$$

2. *Turiyam Neutrosophic Edge Set*: For each edge $(v_i, v_j) \in E$, the Turiyam Neutrosophic graph assigns the following mappings:

$$t(v_i, v_j), iv(v_i, v_j), fv(v_i, v_j), lv(v_i, v_j) : E \to [0, 1],$$

where:

- $t(v_i, v_j)$ is the truth value of the edge (v_i, v_j),
- $iv(v_i, v_j)$ is the indeterminacy value of (v_i, v_j),
- $fv(v_i, v_j)$ is the falsity value of (v_i, v_j),
- $lv(v_i, v_j)$ is the Turiyam Neutrosophic state (or liberal value) of (v_i, v_j),

for all $(v_i, v_j) \in E$, such that the following condition holds for each edge:

$$0 \leq t(v_i, v_j) + iv(v_i, v_j) + fv(v_i, v_j) + lv(v_i, v_j) \leq 4.$$

In this case, V^T represents the Turiyam Neutrosophic vertex set of the graph G^T, and E^T represents the Turiyam Neutrosophic edge set of G^T.

2.6.1 Turiyam Neutrosophic Planar Graph

The Turiyam Neutrosophic planar Graph is an extension of the planar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 131 (Turiyam Neutrosophic Planar Graph)**.** A *Turiyam Neutrosophic planar graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is a Turiyam Neutrosophic graph where:

- $\sigma_T : V \to [0, 1]$, $\sigma_{IV} : V \to [0, 1]$, $\sigma_F : V \to [0, 1]$, and $\sigma_{LV} : V \to [0, 1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each vertex, respectively.
- $\mu_T : E \to [0, 1]$, $\mu_{IV} : E \to [0, 1]$, $\mu_F : E \to [0, 1]$, and $\mu_{LV} : E \to [0, 1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each edge, respectively.

The graph G_T is said to be a *Turiyam Neutrosophic planar graph* if it can be drawn in the plane such that no two edges intersect except at their endpoints.

The Turiyam Neutrosophic planarity value $f_T = (f_T, f_{IV}, f_F, f_{LV})$ of the graph G_T is defined as

$$f_T = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{IV})_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{LV})_{P_i}} \right),$$

where $S_{P_i} = ((S_T)_{P_i}, (S_{IV})_{P_i}, (S_F)_{P_i}, (S_{LV})_{P_i})$ denotes the intersecting values at point P_i. If there are no intersection points, $f_T = (1, 1, 1, 1)$. This value reflects the degree of planarity in the Turiyam Neutrosophic context, with higher values indicating greater planarity.

2.6.2 Turiyam Neutrosophic Outerplanar Graph

The Turiyam Neutrosophic Outerplanar Graph is an extension of the Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 132 (Turiyam Neutrosophic Outerplanar Graph)**.** A *Turiyam Neutrosophic outerplanar graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is a Turiyam Neutrosophic graph that can be embedded in the plane such that all vertices lie on the boundary of the exterior region, satisfying the following conditions:

- $\sigma_T : V \to [0, 1]$, $\sigma_{IV} : V \to [0, 1]$, $\sigma_F : V \to [0, 1]$, and $\sigma_{LV} : V \to [0, 1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each vertex, respectively.
- $\mu_T : E \to [0, 1]$, $\mu_{IV} : E \to [0, 1]$, $\mu_F : E \to [0, 1]$, and $\mu_{LV} : E \to [0, 1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees of each edge, respectively.

- No two edges intersect except at their endpoints, and all vertices are positioned on the outer face of the graph embedding.

If there are intersection points $P_1, P_2, \ldots, P_z$, the Turiyam Neutrosophic outerplanarity value $f_T = (f_T, f_{IV}, f_F, f_{LV})$ of the graph is defined as

$$f_T = \left(\frac{1}{1 + \sum_{i=1}^{z} (S_T)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{IV})_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_F)_{P_i}}, \frac{1}{1 + \sum_{i=1}^{z} (S_{LV})_{P_i}} \right),$$

where $S_{P_i} = ((S_T)_{P_i}, (S_{IV})_{P_i}, (S_F)_{P_i}, (S_{LV})_{P_i})$ denotes the intersecting values at point P_i. If no intersections exist, $f_T = (1, 1, 1, 1)$.

2.6.3 Turiyam Neutrosophic Apex Planar Graph

The Turiyam Neutrosophic Apex planar Graph is an extension of the Apex planar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 133 (Turiyam Neutrosophic Apex Planar Graph)**.** A *Turiyam Neutrosophic Apex Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic planar graph by removing a single vertex $v \in V$. The removed vertex is called an *apex* of the Turiyam Neutrosophic graph ψ_T.

Formally, ψ_T is called a *Turiyam Neutrosophic Apex Planar Graph* if there exists a vertex $v \in V$ such that the induced Turiyam Neutrosophic subgraph $\psi'_T = (V \setminus \{v\}, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$, where $t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$ are the restrictions of $t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$ to the remaining vertices and edges, is a Turiyam Neutrosophic planar graph.

The Turiyam Neutrosophic apex graph retains the following properties:

- $t_A : V \to [0,1]$, $iv_A : V \to [0,1]$, $fv_A : V \to [0,1]$, and $lv_A : V \to [0,1]$ assign the truth, indeterminacy, falsity, and liberal membership degrees to each vertex $v \in V$, respectively.
- $t_B : E \to [0,1]$, $iv_B : E \to [0,1]$, $fv_B : E \to [0,1]$, and $lv_B : E \to [0,1]$ assign the truth, indeterminacy, falsity, and liberal membership degrees to each edge $e \in E$, respectively.
- If the Turiyam Neutrosophic apex graph ψ_T itself is a Turiyam Neutrosophic planar graph, then every vertex in V can be considered an apex.

Similar to classical apex graphs, a Turiyam Neutrosophic apex graph may have multiple apex vertices. Additionally, if ψ_T is the null Turiyam Neutrosophic graph (i.e., $V = \emptyset$), it is trivially a Turiyam Neutrosophic apex graph.

2.6.4 Turiyam Neutrosophic Apex Outerplanar Graph

The Turiyam Neutrosophic Apex Outerplanar Graph is an extension of the Apex Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 134. A *Turiyam Neutrosophic Apex Outerplanar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic outerplanar graph by removing a single vertex $v \in V$.

Formally, ψ_T is called a *Turiyam Neutrosophic Apex Outerplanar Graph* if there exists a vertex $v \in V$ such that the induced Turiyam Neutrosophic subgraph $\psi'_T = (V \setminus \{v\}, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$, where $t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$ are the restrictions of $t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$ to the remaining vertices, is a Turiyam Neutrosophic outerplanar graph.

2.6.5 Turiyam Neutrosophic Quasi-Planar Graph

The Turiyam Neutrosophic Quasi planar Graph is an extension of the Quasi planar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 135. A *Turiyam Neutrosophic Quasi-Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be drawn in the plane such that no three Turiyam Neutrosophic edges are pairwise crossing.

Formally, ψ_T is called a *Turiyam Neutrosophic Quasi-Planar Graph* if there exists a drawing D in which any set of three Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 136. A *Turiyam Neutrosophic k-Quasi-Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be drawn in the plane such that no k Turiyam Neutrosophic edges are pairwise crossing.

Formally, let

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

be a Turiyam Neutrosophic graph where:

- $t_A : V \to [0,1]$, $iv_A : V \to [0,1]$, $fv_A : V \to [0,1]$, and $lv_A : V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal values for each vertex, respectively, satisfying $0 \leq t_A(v) + iv_A(v) + fv_A(v) + lv_A(v) \leq 1$.
- $t_B : V \times V \to [0,1]$, $iv_B : V \times V \to [0,1]$, $fv_B : V \times V \to [0,1]$, and $lv_B : V \times V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal values for each edge (u, v), satisfying $t_B(u,v) + iv_B(u,v) + fv_B(u,v) + lv_B(u,v) \leq 1$ and $t_B(u,v) \leq \min\{t_A(u), t_A(v)\}$.

The Turiyam Neutrosophic graph ψ_T is called a *Turiyam Neutrosophic k-Quasi-Planar Graph* if there exists a drawing D in which any set of k Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

2.6.6 Turiyam Neutrosophic Quasi-Outerplanar Graph

The Turiyam Neutrosophic Quasi Outerplanar Graph is an extension of the Quasi Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 137. A *Turiyam Neutrosophic Quasi-Outerplanar Graph* $\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$ is a Turiyam Neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no three Turiyam Neutrosophic edges are pairwise crossing.

Formally, ψ_T is called a *Turiyam Neutrosophic Quasi-Outerplanar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of three Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

Definition 138. A *Turiyam Neutrosophic Outer-k-Quasi-Planar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be drawn in the plane such that all vertices lie on the boundary of the exterior region, and no set of k Turiyam Neutrosophic edges are pairwise crossing.

Formally, the Turiyam Neutrosophic graph ψ_T is called a *Turiyam Neutrosophic Outer-k-Quasi-Planar Graph* if there exists a drawing D such that all vertices $v \in V$ are positioned on the boundary of the outer (unbounded) face of D, and any set of k Turiyam Neutrosophic edges do not all intersect each other at distinct points in their interiors.

2.6.7 Turiyam Neutrosophic Quasi Apex Graph

The Turiyam Neutrosophic Quasi Apex Graph is an extension of the Quasi Apex Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 139. A *Turiyam Neutrosophic k-Quasi Apex Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic planar graph by removing at most k vertices.

Formally, ψ_T is called a *Turiyam Neutrosophic k-Quasi Apex Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced Turiyam Neutrosophic subgraph $\psi'_T = (V \backslash S, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$, where $t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$ are the restrictions of $t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$ to the remaining vertices, is a Turiyam Neutrosophic quasi planar graph.

2.6.8 Turiyam Neutrosophic Quasi Apex Outerplanar Graph

The Turiyam Neutrosophic Quasi Apex Outerplanar Graph is an extension of the Quasi Apex Outerplanar Graph, incorporating the concepts of Turiyam Neutrosophic Graphs. The definition is provided below.

Definition 140. A *Turiyam Neutrosophic k-Quasi Apex Outerplanar Graph*

$$\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$$

is a Turiyam Neutrosophic graph that can be transformed into a Turiyam Neutrosophic Quasi outerplanar graph by removing at most k vertices.

Formally, ψ_T is called a *Turiyam Neutrosophic k-Quasi Apex Outerplanar Graph* if there exists a subset $S \subseteq V$ with $|S| \leq k$ such that the induced Turiyam Neutrosophic subgraph

$$\psi'_T = (V \setminus S, t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B)$$

, where

$$t'_A, iv'_A, fv'_A, lv'_A, t'_B, iv'_B, fv'_B, lv'_B$$

are the restrictions of

$$t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B$$

to the remaining vertices, is a Turiyam Neutrosophic quasi outerplanar graph.

2.6.9 Turiyam Neutrosophic almost planar graph

The definitions of a Turiyam Neutrosophic almost planar graph and an Turiyam Neutrosophic almost outerplanar graph are extended to the context of Turiyam Neutrosophic graphs as follows.

Definition 141. A *Turiyam Neutrosophic graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is called a *Turiyam Neutrosophic almost planar graph* if there exists an edge $e \in E(G_T)$ such that the Turiyam Neutrosophic graph $G_T - e$ (i.e., the Turiyam Neutrosophic graph obtained by removing the edge e from G_T) is a Turiyam Neutrosophic planar graph.

Definition 142. A *Turiyam Neutrosophic graph* $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ is called a *Turiyam Neutrosophic almost outerplanar graph* if there exists an edge $e \in E(G_T)$ such that the Turiyam Neutrosophic graph $G_T - e$ (i.e., the Turiyam Neutrosophic graph obtained by removing the edge e from G_T) is a Turiyam Neutrosophic outerplanar graph.

Theorem 143. *An Turiyam Neutrosophic almost planar graph can be transformed into an neutrosophic almost planar graph.*

Proof. Obviously holds. □

Theorem 144. *An Turiyam Neutrosophic almost outerplanar graph can be transformed into an neutrosophic almost outerplanar graph.*

Proof. Obviously holds. □

2.7 Plithogenic Graphs and Planar graphs

Recently, Plithogenic Graphs have been proposed as a generalization of Fuzzy Graphs and Turiyam Neutrosophic Graphs, as well as a graphical representation of Plithogenic Sets [4, 264, 449, 488, 491, 496, 506]. Plithogenic Graphs have been developed and are currently being actively studied [239,241,242,321,489,506,511] The definition is provided below.

Definition 145. [511] Let $G = (V, E)$ be a crisp graph where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. A *Plithogenic Graph* PG is defined as:

$$PG = (PM, PN)$$

where:

1. *Plithogenic Vertex Set* $PM = (M, l, Ml, adf, aCf)$:
   - $M \subseteq V$ is the set of vertices.
   - l is an attribute associated with the vertices.
   - Ml is the range of possible attribute values.
   - $adf : M \times Ml \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for vertices.
   - $aCf : Ml \times Ml \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for vertices.
2. *Plithogenic Edge Set* $PN = (N, m, Nm, bdf, bCf)$:
   - $N \subseteq E$ is the set of edges.
   - m is an attribute associated with the edges.
   - Nm is the range of possible attribute values.
   - $bdf : N \times Nm \to [0,1]^s$ is the *Degree of Appurtenance Function (DAF)* for edges.
   - $bCf : Nm \times Nm \to [0,1]^t$ is the *Degree of Contradiction Function (DCF)* for edges.

The Plithogenic Graph PG must satisfy the following conditions:

1. *Edge Appurtenance Constraint*: For all $(x, a), (y, b) \in M \times Ml$:
$$bdf\left((xy), (a, b)\right) \leq \min\{adf(x, a), adf(y, b)\}$$
where $xy \in N$ is an edge between vertices x and y, and $(a, b) \in Nm \times Nm$ are the corresponding attribute values.
2. *Contradiction Function Constraint*: For all $(a, b), (c, d) \in Nm \times Nm$:
$$bCf\left((a, b), (c, d)\right) \leq \min\{aCf(a, c), aCf(b, d)\}$$
3. *Reflexivity and Symmetry of Contradiction Functions*:
$$\begin{aligned} aCf(a, a) &= 0, && \forall a \in Ml \\ aCf(a, b) &= aCf(b, a), && \forall a, b \in Ml \\ bCf(a, a) &= 0, && \forall a \in Nm \\ bCf(a, b) &= bCf(b, a), && \forall a, b \in Nm \end{aligned}$$

Example 146. [235]

- When $s = t = 1$, PG is called a *Plithogenic Fuzzy Graph*.
- When $s = 2, t = 1$, PG is called a *Plithogenic Intuitionistic Fuzzy Graph*.
- When $s = 3, t = 1$, PG is called a *Plithogenic Neutrosophic Graph*.
- When $s = 4, t = 1$, PG is called a *Plithogenic Turiyam Neutrosophic Graph*.

2.7.1 Plithogenic Planar Graph

The Plithogenic Planar Graph is an extension of the Quasi Planar Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 147 (Plithogenic Planar Graph)**.** A *Plithogenic Planar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be embedded in the plane without any edge intersections, except at their endpoints.

Formally, let $PG = (PM, PN)$ be defined as:

- $PM = (M, l, Ml, adf, aCf)$ represents the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$ represents the plithogenic edge set.

For any intersection point P of two plithogenic edges e_1 and e_2, the degree of intersection at P, denoted by

$$S_P = \left(\frac{bdf(e_1) + bdf(e_2)}{2}, \frac{aCf(e_1, e_2) + aCf(e_2, e_1)}{2} \right),$$

must satisfy:

$$S_P = (0, 0) \quad \text{for all points of intersection.}$$

Thus, a Plithogenic Planar Graph PG is defined such that the total number of intersections remains zero.

Theorem 148. *A Plithogenic Planar Graph PG can transform into a Neutrosophic Planar Graph when* $s = 3, t = 1$*, and into a Fuzzy Planar Graph when* $s = 1, t = 1$.

Proof. We prove the theorem in two parts, corresponding to the transformations into a Neutrosophic Planar Graph and a Fuzzy Planar Graph.

A Plithogenic Planar Graph $PG = (PM, PN)$ is defined as:

- $PM = (M, l, Ml, adf, aCf)$, where M is the vertex set, adf is the degree of appurtenance function, and aCf is the degree of contradiction function.
- $PN = (N, m, Nm, bdf, bCf)$, where N is the edge set, bdf is the degree of appurtenance function for edges, and bCf is the degree of contradiction function for edges.

When $s = 3$ and $t = 1$, the Plithogenic Planar Graph transforms into a Neutrosophic Planar Graph $NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3))$, where:

- $\sigma_1, \sigma_2, \sigma_3$ correspond to the truth, indeterminacy, and falsity membership values derived from adf and aCf.
- μ_1, μ_2, μ_3 correspond to the truth, indeterminacy, and falsity membership values derived from bdf and bCf.

The Neutrosophic Planar Graph retains the property that it can be embedded in the plane without any intersecting edges, as $S_P = (0, 0)$ for all intersection points in PG. Thus, the transformation preserves planarity in the neutrosophic context.

When $s = 1$ and $t = 1$, the Plithogenic Planar Graph reduces to a Fuzzy Planar Graph $\psi = (V, \sigma, \mu, E)$, where:

- $\sigma : V \to [0, 1]$ represents the membership degree of each vertex.
- $\mu : E \to [0, 1]$ represents the membership degree of each edge.

The appurtenance and contradiction functions adf, bdf, aCf, and bCf collapse into single membership functions σ and μ. The fuzzy graph satisfies the condition that it can be embedded in the plane without intersecting edges. Specifically, for any intersection point P, the intersection strength I_P satisfies:

$$I_P = 0 \quad \text{for all points of intersection.}$$

This ensures that the fuzzy planarity value $f = 1$, confirming the graph's planarity.

The Plithogenic Planar Graph generalizes both Neutrosophic and Fuzzy Planar Graphs. By adjusting the parameters s and t, the representation transitions into the respective forms, while maintaining the structural property of planarity in all contexts. □

2.7.2 Plithogenic Outerplanar Graph

The Plithogenic Outerplanar Graph is an extension of the Outerplanar Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 149 (Plithogenic Outerplanar Graph)**.** A *Plithogenic Outerplanar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be embedded in the plane such that all vertices lie on the boundary of the exterior region, and there are no intersecting edges except at their endpoints.

Formally, let $PG = (PM, PN)$ be defined as:

- The vertex set $PM = (M, l, Ml, adf, aCf)$ has every vertex $m \in M$ located on the outer boundary.
- The edge set $PN = (N, m, Nm, bdf, bCf)$ ensures that any intersection S_P between edges satisfies $S_P = (0, 0)$, indicating no intersections.

Therefore, all vertices are on the outer face, making PG an outerplanar graph in the plithogenic context.

Theorem 150. *A Plithogenic Outerplanar Graph can transform into a Neutrosophic Outerplanar Graph when* $s = 3, t = 1$*, and into a Fuzzy Outerplanar Graph when* $s = 1, t = 1$.

Proof. A Plithogenic Outerplanar Graph $PG = (PM, PN)$ is defined as:

- $PM = (M, l, Ml, adf, aCf)$, where M is the vertex set, adf is the degree of appurtenance function, and aCf is the degree of contradiction function.
- $PN = (N, m, Nm, bdf, bCf)$, where N is the edge set, bdf is the degree of appurtenance function for edges, and bCf is the degree of contradiction function for edges.

By setting $s = 3$ and $t = 1$, the Plithogenic Outerplanar Graph becomes a Neutrosophic Outerplanar Graph:

$$NTG = (V, E, \sigma = (\sigma_1, \sigma_2, \sigma_3), \mu = (\mu_1, \mu_2, \mu_3)),$$

where:

- $\sigma_1, \sigma_2, \sigma_3$ correspond to the membership, indeterminacy, and contradiction values derived from adf and aCf.
- μ_1, μ_2, μ_3 correspond to the membership, indeterminacy, and contradiction values derived from bdf and bCf.

The Neutrosophic Outerplanar Graph inherits the condition that all vertices lie on the boundary of the outer face, and no edges intersect except at their endpoints. Thus, the transformation preserves the outerplanarity property.

When $s = 1$ and $t = 1$, the Plithogenic Outerplanar Graph reduces to a Fuzzy Outerplanar Graph:

$$\psi = (V, \sigma, \mu),$$

where:

- $\sigma : V \to [0, 1]$ represents the membership degree of each vertex.
- $\mu : E \to [0, 1]$ represents the membership degree of each edge.

The appurtenance and contradiction functions adf, bdf, aCf, and bCf collapse into a single membership function for vertices and edges. The condition of outerplanarity remains satisfied because the structural constraints of Plithogenic Outerplanar Graphs ensure that:

$$i(\psi) = 0,$$

indicating that all vertices are on the boundary of the outer face, with no intersecting edges.

The Plithogenic Outerplanar Graph generalizes both Neutrosophic and Fuzzy Outerplanar Graphs. By adjusting the parameters s and t, the graph's representation transitions into the corresponding forms while preserving the structural property of outerplanarity. □

2.7.3 Plithogenic Apex Graph

The Plithogenic Apex Graph is an extension of the Apex Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 151 (Plithogenic Apex Planar Graph)**.** A *Plithogenic Apex Planar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be made a Plithogenic Planar Graph by removing a single vertex $v \in M$.

Formally, PG is called a *Plithogenic Apex Planar Graph* if there exists a vertex $v \in M$ such that the induced subgraph $PG' = (PM', PN')$, where:

- $PM' = (M \setminus \{v\}, l', Ml', adf', aCf')$
- $PN' = (N', m', Nm', bdf', bCf')$

is a Plithogenic Planar Graph. The functions adf', aCf', bdf', bCf' are the restrictions of

$$adf, aCf, bdf, bCf$$

to the remaining vertices and edges, ensuring no edge crossings in the induced subgraph.

2.7.4 Plithogenic Apex Outerplanar Graph

The Plithogenic Apex Outerplanar Graph is an extension of the Apex Outerplanar Graph, incorporating the concepts of Plithogenic Graphs. The definition is provided below.

Definition 152 (Plithogenic Apex Outerplanar Graph)**.** A *Plithogenic Apex Outerplanar Graph* $PG = (PM, PN)$ is a plithogenic graph that can be made a Plithogenic Outerplanar Graph by removing a single vertex $v \in M$.

Formally, PG is called a *Plithogenic Apex Outerplanar Graph* if there exists a vertex $v \in M$ such that the induced subgraph $PG' = (PM', PN')$, where:

- $PM' = (M \setminus \{v\}, l', Ml', adf', aCf')$
- $PN' = (N', m', Nm', bdf', bCf')$

is a Plithogenic Outerplanar Graph. The functions adf', aCf', bdf', bCf' are the restrictions of

$$adf, aCf, bdf, bCf$$

to the remaining vertices and edges, ensuring the subgraph has all vertices lying on the boundary of the exterior region with no edge intersections.

Theorem 153. *A Plithogenic Apex Planar Graph PG with* $s = 4, t = 1$ *can transform into a Turiyam Neutrosophic Apex Planar Graph, and a Plithogenic Apex Outerplanar Graph PG with* $s = 4, t = 1$ *can transform into a Turiyam Neutrosophic Apex Outerplanar Graph.*

Proof. We prove both cases separately.

Case 1: Transformation to a Turiyam Neutrosophic Apex Planar Graph A Plithogenic Apex Planar Graph $PG = (PM, PN)$ is defined as:

- $PM = (M, l, Ml, adf, aCf)$, representing the plithogenic vertex set.
- $PN = (N, m, Nm, bdf, bCf)$, representing the plithogenic edge set.

When $s = 4$ and $t = 1$, the Plithogenic Apex Planar Graph transforms into a Turiyam Neutrosophic Apex Planar Graph $\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$, where:

- The attributes l, adf, aCf of vertices in PM are mapped to the Turiyam Neutrosophic truth, indeterminacy, falsity, and liberal membership degrees t_A, iv_A, fv_A, lv_A, respectively.
- The attributes m, bdf, bCf of edges in PN are mapped to the Turiyam Neutrosophic truth, indeterminacy, falsity, and liberal membership degrees t_B, iv_B, fv_B, lv_B, respectively.

By definition, a Plithogenic Apex Planar Graph becomes a Plithogenic Planar Graph when a single vertex $v \in M$ is removed. This property carries over to the Turiyam Neutrosophic domain, where removing a single vertex $v \in V$ results in a Turiyam Neutrosophic Planar Graph ψ'_T. Therefore, the transformation preserves the apex property.

Case 2: Transformation to a Turiyam Neutrosophic Apex Outerplanar Graph Similarly, a Plithogenic Apex Outerplanar Graph $PG = (PM, PN)$ satisfies the condition that removing a single vertex $v \in M$ results in a Plithogenic Outerplanar Graph PG'. When $s = 4$ and $t = 1$, the Plithogenic Apex Outerplanar Graph transforms into a Turiyam Neutrosophic Apex Outerplanar Graph $\psi_T = (V, t_A, iv_A, fv_A, lv_A, t_B, iv_B, fv_B, lv_B)$ by mapping the plithogenic membership functions adf, aCf, bdf, bCf to the Turiyam Neutrosophic dimensions.

The outerplanar property is preserved under this transformation because removing a single vertex $v \in V$ results in a Turiyam Neutrosophic Outerplanar Graph ψ'_T, where all remaining vertices lie on the boundary of the exterior face, and no edge intersections occur.

Both transformations rely on the mapping of plithogenic attributes to Turiyam Neutrosophic dimensions. The structural properties (planarity and outerplanarity) and the apex property are preserved during these transformations, completing the proof. □

3. Result of this paper

3.1 Some property of Turiyam Neutrosophic Planar Graph

We will examine the properties of a Turiyam Neutrosophic Planar Graph. Similar characteristics can also be observed in other graphs such as fuzzy planar graphs, Neutrosophic planar graphs, and Plithogenic Planar Graphs.

Theorem 154. *If a Turiyam Neutrosophic graph G_T is homomorphic to the complete graph K_4, then G_T is a Turiyam Neutrosophic planar graph.*

Proof. Let $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ be a Turiyam Neutrosophic graph, where:

- $\sigma_T : V \to [0,1]$, $\sigma_{IV} : V \to [0,1]$, $\sigma_F : V \to [0,1]$, and $\sigma_{LV} : V \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees for each vertex $v \in V$.
- $\mu_T : E \to [0,1]$, $\mu_{IV} : E \to [0,1]$, $\mu_F : E \to [0,1]$, and $\mu_{LV} : E \to [0,1]$ represent the truth, indeterminacy, falsity, and liberal membership degrees for each edge $e \in E$.

Assume G_T is homomorphic to the complete graph K_4. This means there exists a mapping $\phi : V(G_T) \to V(K_4)$ such that if $(u, v) \in E(G_T)$, then $(\phi(u), \phi(v)) \in E(K_4)$.

Since K_4 is planar, this homomorphism implies that G_T inherits a planar structure. Thus, G_T can be embedded in the plane in a manner where no two edges intersect except at their endpoints.

Therefore, G_T is a Turiyam Neutrosophic planar graph by definition. Additionally, the Turiyam Neutrosophic planarity value $f_T = (1, 1, 1, 1)$, indicating the highest degree of planarity. □

Theorem 155. *If a Turiyam Neutrosophic graph G_T is homomorphic to a tree graph, then G_T is a Turiyam Neutrosophic planar graph.*

Proof. The proof is analogous to the method used in the previous proof. □

Theorem 156. *If a Turiyam Neutrosophic graph G_T is homomorphic to a cycle graph, grid graph, or wheel graph, then G_T is a Turiyam Neutrosophic planar graph.*

Proof. Let $G_T = (V, E, \sigma_T, \sigma_{IV}, \sigma_F, \sigma_{LV})$ be a Turiyam Neutrosophic graph, where:

- V is the set of vertices, and E is the set of edges.
- σ_T, σ_{IV}, σ_F, and σ_{LV} represent the truth, indeterminacy, falsity, and liberal membership degrees for vertices and edges.

We consider three cases:

1. *Cycle Graph:* A cycle graph is planar since it can be drawn as a simple closed loop without intersecting edges[431, 521, 536]. If G_T is homomorphic to a cycle graph, then a mapping exists that preserves the structure, making G_T planar.
2. *Grid Graph:* A grid graph forms a rectangular lattice and is also planar. The homomorphism from G_T to the grid graph implies G_T inherits the planar characteristics, ensuring no edge intersections[278, 294, 306, 424].
3. *Wheel Graph:* A wheel graph consists of a central vertex connected to all vertices of an outer cycle[354, 504]. This graph can be drawn without any intersecting edges. Therefore, a homomorphism from G_T to a wheel graph means G_T is also planar.

In all cases, G_T can be embedded in the plane without intersecting edges, confirming that G_T is a Turiyam Neutrosophic planar graph. □

Theorem 157. *Let G_T be a Turiyam Neutrosophic graph. If G_T is homomorphic to any of the following graph classes:*

- *Halin graphs (also called skirted trees or roofless polyhedra),*
- *Butterfly graphs (also known as bowtie or hourglass graphs),*
- *Herschel graph,*
- *Polyhedral graph,*
- *Golomb graph,*
- *Tutte graph,*
- *Bull graph,*
- *Goldner–Harary graph,*
- *Nested triangles graph,*
- *Frucht graph,*
- *Friendship graphs,*
- *26-fullerene graph,*
- *Antiprism graph,*
- *Archimedean graph,*
- *Wiener–Araya graph,*
- *Araya–Wiener graph,*
- *Goddard–Henning graph,*
- *Heawood Four-Color graph,*
- *Kittell graph,*
- *Poussin graph,*
- *Cactus graph,*
- *Fritsch graph,*
- *Soifer graph,*
- *Ladder graphs,*

then G_T is a Turiyam Neutrosophic planar graph.

Proof. The proof follows by leveraging the planar properties of the listed graph classes.

Planarity of Listed Graph Classes:

- *Halin graphs* are known to be planar as they are constructed by adding a cycle to the leaves of a rooted tree [113, 167, 270].
- *Butterfly graphs* (or bowtie/hourglass graphs) are planar due to their simple structure of two triangles sharing a common vertex [302, 303].
- *The Herschel graph*, being a planar graph, has been studied in various applications [508].
- *Polyhedral graphs* represent the skeletons of convex polyhedra, which are planar by definition [81, 340].
- *Golomb graphs* are planar due to their simple combinatorial construction [500].
- *Tutte graphs*, as described in graph theory literature, are planar [542].
- *Bull graphs*, representing a triangle with two disjoint pendant edges, are planar [162, 359].
- *Goldner–Harary graphs* are maximal planar graphs and thus planar by construction [262].
- *Nested triangles graphs* have a planar embedding due to their layered triangular structure [229].
- *Frucht graphs* are planar as they are minimal vertex-transitive graphs [438].
- *Friendship graphs* (or windmill graphs) are planar since they consist of triangles sharing a common vertex [47, 535].
- *The 26-fullerene graph* is planar by definition as a fullerene graph [272].
- *Antiprism graphs* are planar due to their combinatorial structure [288, 513].
- *Archimedean graphs* are planar since they represent tilings of the Euclidean plane [225].
- *Wiener–Araya graphs* and *Araya–Wiener graphs* are planar as they extend planar cubic graphs [368, 480].
- *Goddard–Henning graphs* maintain planarity as part of their structural properties [261].
- *The Heawood Four-Color graph* is planar due to its association with the Four-Color Theorem [253].
- *Kittell graphs* and *Poussin graphs* are planar due to their simple construction [339, 546].
- *Cactus graphs*, composed of edge-disjoint cycles, are planar [91, 433].
- *Fritsch and Soifer graphs* are planar examples studied in the context of coloring problems [253].
- *Ladder graphs*, as simple planar grid graphs, are also planar [357, 402].

Since G_T is homomorphic to these graph classes, it inherits their planarity. Hence, G_T is a Turiyam Neutrosophic planar graph. For more detailed references and properties of each graph class, see the cited works. □

Theorem 158. *If a Turiyam Neutrosophic graph G_T is homomorphic to a complete graph K_5, then G_T is a Non-Turiyam Neutrosophic planar graph.*

Proof. We will prove this by contradiction. Assume, to the contrary, that a Turiyam Neutrosophic graph G_T is homomorphic to the complete graph K_5, but G_T is a Turiyam Neutrosophic planar graph.

By definition, a Turiyam Neutrosophic planar graph can be embedded in the plane such that no two edges intersect except at their endpoints. However, the complete graph K_5 is known to be non-planar, as it is impossible to draw K_5 in the plane without intersecting edges.

If G_T is homomorphic to K_5, there exists a mapping $\phi : V(G_T) \to V(K_5)$ such that for every edge $(u, v) \in E(G_T)$, the edge $(\phi(u), \phi(v)) \in E(K_5)$. This implies that the structure of G_T must follow that of K_5.

Since K_5 is non-planar, the graph G_T, being homomorphic to K_5, must also be non-planar. This contradicts our assumption that G_T is a Turiyam Neutrosophic planar graph.

Therefore, our assumption is false, and we conclude that G_T is a Non-Turiyam Neutrosophic planar graph. □

Theorem 159. *If a Turiyam Neutrosophic graph G_T is homomorphic to the complete bipartite graph $K_{3,3}$, then G_T is a Non-Turiyam Neutrosophic planar graph.*

Proof. We will prove this by contradiction. Assume, to the contrary, that a Turiyam Neutrosophic graph G_T is homomorphic to the complete bipartite graph $K_{3,3}$, but G_T is a Turiyam Neutrosophic planar graph.

By definition, a Turiyam Neutrosophic planar graph can be embedded in the plane such that no two edges intersect except at their endpoints. However, it is well-known from Kuratowski's Theorem that the complete bipartite graph $K_{3,3}$ is non-planar, meaning it is impossible to draw $K_{3,3}$ in the plane without intersecting edges.

If G_T is homomorphic to $K_{3,3}$, there exists a mapping $\phi : V(G_T) \to V(K_{3,3})$ such that for every edge $(u, v) \in E(G_T)$, the edge $(\phi(u), \phi(v)) \in E(K_{3,3})$. This implies that G_T inherits the structural relationships of $K_{3,3}$ and must follow its non-planar nature.

Since $K_{3,3}$ is non-planar, the graph G_T, being homomorphic to $K_{3,3}$, must also be non-planar. This contradicts our assumption that G_T is a Turiyam Neutrosophic planar graph.

Therefore, our assumption is false, and we conclude that G_T is a Non-Turiyam Neutrosophic planar graph. □

3.2 Some property of Neutrosophic outerplanar graph

We will examine some properties of the Neutrosophic outerplanar graph. It should be noted that the same properties hold when Neutrosophic is replaced with Fuzzy or Plithogenic. The following theorem holds.

Theorem 160. *A neutrosophic outerplanar graph $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is a neutrosophic planar graph.*

Proof. By definition, a neutrosophic outerplanar graph $G = (A, B, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ is embedded in the plane such that all vertices lie on the boundary of the exterior region, and no two edges intersect except at their endpoints.

We verify that G satisfies the conditions of a neutrosophic planar graph:

1. *Verification of Intersection Properties:* By definition, the neutrosophic outerplanar graph does not have edge crossings within the interior of the graph. Thus, the graph satisfies the conditions for planarity.
2. *Calculation of Neutrosophic Planarity Value:* For a neutrosophic planar graph, we consider the intersection points $P_1, P_2, \ldots, P_z$ between edges, and the neutrosophic planarity value $f = (f_T, f_I, f_F)$ is given by

$$f = \left(\frac{1}{1 + \{(S_T)_{P_1} + (S_T)_{P_2} + \ldots + (S_T)_{P_z}\}}, \right.$$
$$\frac{1}{1 + \{(S_I)_{P_1} + (S_I)_{P_2} + \ldots + (S_I)_{P_z}\}},$$
$$\left. \frac{1}{1 + \{(S_F)_{P_1} + (S_F)_{P_2} + \ldots + (S_F)_{P_z}\}} \right).$$

Since a neutrosophic outerplanar graph has no intersections except at endpoints, $(S_T)_{P_i} = 0$, $(S_I)_{P_i} = 0$, and $(S_F)_{P_i} = 0$ for all P_i. Therefore,

$$f = \left(\frac{1}{1+0}, \frac{1}{1+0}, \frac{1}{1+0} \right) = (1, 1, 1).$$

This value $f = (1, 1, 1)$ confirms that the graph is completely planar.

Since the neutrosophic outerplanar graph satisfies the conditions of a neutrosophic planar graph, we conclude that every neutrosophic outerplanar graph is indeed a neutrosophic planar graph. □

Theorem 161. *A neutrosophic graph G is a neutrosophic outerplanar graph if and only if it contains no neutrosophic subgraph homomorphic to K_4 or $K_{2,3}$.*

Proof. First, note that K_4 and $K_{2,3}$ are not neutrosophic outerplanar graphs, as their structure does not allow all vertices to lie on the boundary of an exterior face without crossings, even when considering neutrosophic parameters.

(Necessity) Suppose G is a neutrosophic graph that is outerplanar. Then, by definition, G can be embedded in a plane such that all vertices lie on the boundary of a single unbounded face without intersecting edges. If G contained a subgraph homomorphic to K_4 or $K_{2,3}$, such subgraphs would inherently require intersecting edges in any planar embedding, contradicting the outerplanar property. Therefore, G contains no neutrosophic subgraph homomorphic to K_4 or $K_{2,3}$.

(Sufficiency) Conversely, assume G is a neutrosophic graph containing no subgraph homomorphic to K_4 or $K_{2,3}$. By Kuratowski's Theorem, since G does not have a K_4 or $K_{2,3}$ subgraph, G must be planar. Assume for contradiction that G is not a neutrosophic outerplanar graph.

Let G be embedded such that it is a cyclic block not satisfying the outerplanar property. Suppose the embedding is such that the exterior region contains the maximum number of vertices. Let C be the boundary cycle of this exterior region. Since G is not outerplanar, there exists at least one vertex d lying in the interior of C.

If there exists a vertex d in the interior connected by three mutually disjoint paths to three distinct vertices on C, then G contains a subgraph homomorphic to K_4. Alternatively, if d has two disjoint paths connecting it to two disjoint vertices a and e on C, and the edge ae is not on C, then G contains a subgraph homomorphic to $K_{2,3}$. Both cases lead to contradictions since we assumed G contained no such subgraphs.

Thus, G is a neutrosophic outerplanar graph, completing the proof. □

Theorem 162. *If ψ is a connected neutrosophic outerplanar graph, then it has a neutrosophic dual graph ψ^* such that there exists a vertex v for which $\psi^* - v$ contains no neutrosophic cycle.*

Proof. Let $\psi = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a connected neutrosophic outerplanar graph with no intersections of edges $E' \subseteq E$ within the graph. We can divide the neutrosophic graph ψ into a finite number of regions. Each of these regions corresponds to a neutrosophic face $F_1, F_2, F_3, \ldots, F_k$.

For each face F_j, its membership value is determined by

$$F_j = \min\left\{ \frac{\mu_T(x,y)}{\sigma_T(x) \wedge \sigma_T(y)}, \frac{\mu_I(x,y)}{\sigma_I(x) \wedge \sigma_I(y)}, \frac{\mu_F(x,y)}{\sigma_F(x) \wedge \sigma_F(y)} \mid \text{All edges bounding the region} \right\}.$$

When dividing the neutrosophic graph into such regions with neutrosophic face values, we construct the neutrosophic dual graph $\psi^* = (V^*, \sigma^*, \mu^*)$ as follows:

- Each face of ψ corresponds to a vertex in ψ^*.
- Each edge of ψ corresponds to an edge in ψ^*.

Since ψ is an outerplanar graph, the outer face forms a neutrosophic cycle. Therefore, the neutrosophic dual graph ψ^* exists for ψ.

Now, consider the removal of a vertex v from ψ^*. This corresponds to removing a face of the original graph ψ. The remaining graph $\psi^* - v$ represents the connections among the other faces of ψ but without forming any cycles since the outer face is removed.

Thus, $\psi^* - v$ contains no neutrosophic cycle, completing the proof. □

Theorem 163. *A neutrosophic outerplanar graph ψ always has a vertex-deleted neutrosophic outerplanar subgraph that is also a vertex-deleted neutrosophic subgraph of ψ.*

Proof. Let ψ be a neutrosophic outerplanar graph, and let H be a subgraph obtained by deleting a vertex from ψ. Since ψ is outerplanar, all vertices lie on the outer face in the neutrosophic embedding of the graph. Thus, removing any vertex $v \in V(\psi)$ from ψ does not affect the outerplanarity property of the graph. Therefore, the resulting subgraph $\psi - v$ is still a neutrosophic outerplanar graph.

Furthermore, because $H = \psi - v$ retains all other vertices and edges apart from the deleted vertex, H is a valid vertex-deleted neutrosophic subgraph of ψ. Thus, every neutrosophic outerplanar graph has a vertex-deleted subgraph that maintains its outerplanarity property and is also a valid subgraph of the original graph ψ, as required. □

Definition 164. A *Neutrosophic Forest* is a collection of one or more disjoint neutrosophic trees. Formally, a neutrosophic forest F is defined as a pair $F = (V, E)$, where:

- V is a finite set of vertices.
- $E \subseteq V \times V$ is a set of edges.

Each vertex $v \in V$ and each edge $e \in E$ is associated with neutrosophic membership values:

- The vertex membership function $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ assigns to each vertex v:
  - $\sigma_T(v) \in [0, 1]$: Truth membership degree.
  - $\sigma_I(v) \in [0, 1]$: Indeterminacy membership degree.
  - $\sigma_F(v) \in [0, 1]$: Falsity membership degree.
- The edge membership function $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ assigns to each edge $e = (u, v) \in E$:
  - $\mu_T(e) \in [0, 1]$: Truth membership degree.
  - $\mu_I(e) \in [0, 1]$: Indeterminacy membership degree.
  - $\mu_F(e) \in [0, 1]$: Falsity membership degree.

A neutrosophic forest satisfies the following properties:

1. Each connected component is a neutrosophic tree with no cycles.
2. The forest consists of disjoint neutrosophic trees.
3. For any vertex v, $\sigma_T(v) + \sigma_I(v) + \sigma_F(v) \leq 1$, and for any edge e, $\mu_T(e) + \mu_I(e) + \mu_F(e) \leq 1$.

Theorem 165. *A Neutrosophic Forest is a Neutrosophic graph.*

Proof. Obviously holds. □

Theorem 166. *A neutrosophic forest is a neutrosophic outerplanar graph.*

Proof. Let $F = (V, E, \sigma = (\sigma_T, \sigma_I, \sigma_F), \mu = (\mu_T, \mu_I, \mu_F))$ be a neutrosophic forest, where:

- V is the set of vertices.
- E is the set of edges.
- $\sigma(v) = (\sigma_T(v), \sigma_I(v), \sigma_F(v))$ represents the truth, indeterminacy, and falsity membership degrees for each vertex $v \in V$.
- $\mu(e) = (\mu_T(e), \mu_I(e), \mu_F(e))$ represents the truth, indeterminacy, and falsity membership degrees for each edge $e \in E$.

By definition, a neutrosophic forest is a collection of one or more disjoint neutrosophic trees, where each tree is acyclic and connected. This means that the overall structure of F contains no cycles.

To prove that F is a neutrosophic outerplanar graph, we need to show that:

1. F can be embedded in the plane such that all vertices lie on the boundary of the exterior region.
2. No two edges intersect except at their endpoints.

Since F consists of disjoint trees, each tree can be embedded in the plane without any cycles. All vertices of each tree can be arranged along a straight line or a circle, ensuring that every vertex lies on the boundary of the exterior face. Thus, no edge crossings occur, and all vertices lie on the outer face.

Therefore, F satisfies the conditions for being a neutrosophic outerplanar graph. Hence, every neutrosophic forest is a neutrosophic outerplanar graph. □

Corollary 167. *A neutrosophic tree is a neutrosophic outerplanar graph.*

Proof. Obviously holds. □

Theorem 168. *Let ψ be a connected neutrosophic outerplanar graph such that $f_\psi \neq 1$, and let W be a subset of its vertices such that $W \subseteq V$. The neutrosophic outerplanar subgraph of ψ obtained by deleting the vertices in W is $\psi' = \psi - W$. Its neutrosophic dual graph is denoted as ψ''.*

Proof. Consider a connected neutrosophic outerplanar graph $\psi = (V, E, \sigma, \mu)$ and a subset of vertices $W \subseteq V$. Let $\psi - W$ be the subgraph obtained by deleting the vertices in W, which we denote as ψ'. By construction, ψ' is a vertex-deleted neutrosophic outerplanar subgraph of ψ.

Since ψ is a neutrosophic outerplanar graph, the removal of vertices in W does not introduce any intersections or additional complexity that would violate the outerplanar property. Hence, ψ' retains the neutrosophic outerplanarity.

Next, we divide the neutrosophic graph ψ' into a finite number of regions, where each region corresponds to a neutrosophic face. The membership value for each face F_j is determined by

$$F_j = \min \left\{ \frac{\mu_T(x,y)}{\sigma_T(x) \wedge \sigma_T(y)}, \frac{\mu_I(x,y)}{\sigma_I(x) \wedge \sigma_I(y)}, \frac{\mu_F(x,y)}{\sigma_F(x) \wedge \sigma_F(y)} \mid \text{all edges bounding the region} \right\}.$$

Since each neutrosophic outerplanar graph with a planarity value $f_\psi \neq 1$ has a corresponding neutrosophic dual graph, there exists a dual graph ψ'' for the vertex-deleted subgraph ψ'. In this dual graph ψ'', each face of the original graph ψ' corresponds to a vertex, and each edge in ψ' corresponds to an edge in ψ''.

Therefore, the neutrosophic dual graph ψ'' exists for the vertex-deleted neutrosophic outerplanar subgraph $\psi' = \psi - W$. This completes the proof. □

4. Conclusion and Future Works

This section presents the conclusion and outlines future works.

4.1 Conclusion in this paper

In this paper, we investigated outerplanar graphs and aim to clarify the relationships between planar graphs, fuzzy graphs, and neutrosophic graphs.

Given the extensive and continuously expanding body of literature on fuzzy mathematics, it is inevitable that similar concepts may emerge independently across different journals and periods. Nevertheless, we believe that efforts to unify these concepts are essential and will greatly contribute to advancing the field. We aim to conduct further investigations in this direction in the future(cf.[235]).

4.2 Future works: planar hypergraphs

In the future, we will explore planar hypergraphs. A hypergraph is a generalization of a graph in which edges, known as hyperedges, can connect any number of vertices, not just two. The formal definition of a hypergraph is given below[127].

Definition 169. [127] A hypergraph is a pair $H = (V(H), E(H))$, consisting of a nonempty set $V(H)$ of vertices and a set $E(H)$ of subsets of $V(H)$, referred to as the hyperedges of H. In this context, we consider only finite hypergraphs.

This hypergraph structure is valuable for modeling complex relationships in various fields, such as computer science and biology [221, 266, 267, 437]. Research on databases[44, 217, 304, 541] using hypergraphs and Hypergraph neural networks [133, 155, 205, 222, 249, 313, 364] are also actively being conducted. A planar hypergraph can be seen as the hypergraph counterpart of a planar graph, and its definition is provided below. It is important to note that a planar graph is a special case of a planar hypergraph where all edges consist of exactly two vertices.

Definition 170. (cf.[180, 208, 347, 545]) Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of hyperedges (subsets of V).

The bipartite representation of H, denoted by $B(H)$, is a bipartite graph with a vertex set $V \cup E$. In $B(H)$, a vertex $v \in V$ is adjacent to a vertex $e \in E$ if and only if $v \in e$ in the original hypergraph H.

A hypergraph H is called a *planar hypergraph* if its bipartite representation $B(H)$ is a planar graph. In other words, H is a planar hypergraph if $B(H)$ can be drawn on a plane without any edge crossings.

Theorem 171. *A planar hypergraph $H = (V, E)$ can be transformed into a planar graph by representing it as its bipartite graph $B(H)$.*

Proof. The bipartite representation $B(H)$ of a planar hypergraph H is planar by definition. $B(H)$ is constructed by treating V and E as two disjoint vertex sets and connecting a vertex $v \in V$ to an edge vertex $e \in E$ if and only if $v \in e$. Since H is planar, $B(H)$ can be drawn on a plane without edge crossings, satisfying the planar graph property. □

Definition 172 (Outerplanar Hypergraph)**.** Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of hyperedges (subsets of V). The bipartite representation of H, denoted by $B(H)$, is a bipartite graph with vertex set $V \cup E$. In $B(H)$, a vertex $v \in V$ is adjacent to a vertex $e \in E$ if and only if $v \in e$ in the original hypergraph H.

A hypergraph H is called an *outerplanar hypergraph* if its bipartite representation $B(H)$ is an outerplanar graph. In other words, H is an outerplanar hypergraph if $B(H)$ can be embedded in the plane such that all vertices lie on the boundary of the exterior face, and no two edges intersect except at their endpoints.

Theorem 173. *An outerplanar hypergraph $H = (V, E)$ can be transformed into an outerplanar graph by representing it as its bipartite graph $B(H)$.*

Proof. By definition, an outerplanar hypergraph H has a bipartite representation $B(H)$ that is outerplanar. $B(H)$ can be embedded in the plane such that all vertices lie on the boundary of the outer face, and no two edges intersect except at their endpoints. Hence, $B(H)$ is an outerplanar graph. □

Definition 174 (Apex Hypergraph)**.** A hypergraph $H = (V, E)$ is called an *apex hypergraph* if it can be made planar by removing a single vertex $v \in V$ from H. Formally, H is an apex hypergraph if there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$, where $H \setminus \{v\}$ denotes the hypergraph obtained by removing v and all hyperedges containing v, is a planar graph.

The properties of an apex hypergraph include:

- If the hypergraph H itself is planar, then any vertex can be considered an apex.

Theorem 175. *An apex hypergraph $H = (V, E)$ can be transformed into a planar hypergraph by removing a single vertex $v \in V$.*

Proof. By definition, an apex hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$ is planar. Removing the vertex v and all hyperedges containing v eliminates all crossings introduced by v. The resulting hypergraph $H \setminus \{v\}$ is planar by the property of its bipartite representation. Hence, $H \setminus \{v\}$ is a planar hypergraph. □

Theorem 176. *An apex hypergraph $H = (V, E)$ can be transformed into an apex graph by representing it as its bipartite graph $B(H)$.*

Proof. An apex hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the hypergraph $H \setminus \{v\}$, obtained by removing v and all hyperedges incident to v, is a planar hypergraph. The bipartite representation $B(H)$ of H is a graph with vertex sets $V \cup E$, where a vertex $v \in V$ is adjacent to $e \in E$ if $v \in e$. Removing v from $B(H)$ reduces the bipartite graph to a representation of $H \setminus \{v\}$, which is a planar graph. Thus, $B(H)$ can be transformed into an apex graph by removing the vertex v. □

Definition 177 (Apex Outerplanar Hypergraph)**.** A hypergraph $H = (V, E)$ is called an *apex outerplanar hypergraph* if it can be made outerplanar by removing a single vertex $v \in V$ from H. Formally, H is an apex outerplanar hypergraph if there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$, where $H \setminus \{v\}$ denotes the hypergraph obtained by removing v and all hyperedges containing v, is an outerplanar graph.

The properties of an apex outerplanar hypergraph include:

- If the hypergraph H itself is outerplanar, then any vertex can be considered an apex.

Theorem 178. *An apex outerplanar hypergraph $H = (V, E)$ can be transformed into an outerplanar hypergraph by removing a single vertex $v \in V$.*

Proof. By definition, an apex outerplanar hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the bipartite representation $B(H \setminus \{v\})$ is outerplanar. Removing v and its associated hyperedges reduces $B(H)$ to an outerplanar graph. The resulting hypergraph $H \setminus \{v\}$ retains the outerplanar property in its bipartite representation, making it an outerplanar hypergraph. □

Theorem 179. *An apex outerplanar hypergraph $H = (V, E)$ can be transformed into an apex outerplanar graph by representing it as its bipartite graph $B(H)$.*

Proof. By definition, an apex outerplanar hypergraph H is a hypergraph for which there exists a vertex $v \in V$ such that the hypergraph $H \setminus \{v\}$, obtained by removing v and all hyperedges incident to v, is an outerplanar hypergraph. The bipartite representation $B(H)$ of H is a graph with vertex sets $V \cup E$, where a vertex $v \in V$ is adjacent to $e \in E$ if $v \in e$. Removing v from $B(H)$ reduces the bipartite graph to a representation of $H \setminus \{v\}$, which is an outerplanar graph. Thus, $B(H)$ can be transformed into an apex outerplanar graph by removing the vertex v. □

I anticipate that future research will increasingly apply concepts such as planar hypergraphs, outerplanar hypergraphs, apex hypergraphs, and apex outerplanar hypergraphs to structures like fuzzy hypergraphs and neutrosophic hypergraphs. Furthermore, we plan to investigate planar hypergraph neural networks, which are hypergraph neural networks constructed based on planar hypergraphs. In addition, we intend to explore superhypergraph neural networks by applying the concept of superhypergraphs to neural networks.

4.3 Future works: planar and outerplanar superhypergraph

We will also explore planar, outerplanar, apex, and apex outerplanar superhypergraphs. Superhypergraphs [236, 240, 254, 280, 281, 492–495, 498] are well-established graph classes that bridge the concepts of graphs and hypergraphs. Future studies aim to investigate the mathematical structures and practical applications of planar, outerplanar, apex, and apex outerplanar superhypergraphs in greater depth.

Definition 180 (Extended Bipartite Representation of a SuperHypergraph)**.** The *Extended Bipartite Representation* of a superhypergraph SHG, denoted $B(SHG)$, is a bipartite graph (U, F) where:

- $U = V \cup E$, treating hyperedges as vertices.
- An edge $(u, e) \in F$ exists if:
  - $u \in V$ and $u \in e$ in SHG.
  - $u \in E$ and there is a higher-order relationship between hyperedge u and hyperedge e.

Definition 181 (Planar SuperHypergraph)**.** A *Planar SuperHypergraph SHG* is a superhypergraph whose extended bipartite representation $B(SHG)$ is a planar graph.

Definition 182 (Outerplanar SuperHypergraph)**.** An *Outerplanar SuperHypergraph SHG* is a superhypergraph whose extended bipartite representation $B(SHG)$ is an outerplanar graph.

Theorem 183. *Planar superhypergraphs and outerplanar superhypergraphs generalize planar hypergraphs and outerplanar hypergraphs, respectively.*

Proof. We will demonstrate that:

1. *Every Planar Hypergraph is a Planar SuperHypergraph.*

   A hypergraph is a special case of a superhypergraph where hyperedges are solely subsets of V, without any higher-order connections. For a planar hypergraph $H = (V, E)$, its bipartite representation $B(H)$ is planar.

   Viewing H as a superhypergraph SHG without superedges, the extended bipartite representation $B(SHG)$ is identical to $B(H)$. Therefore, SHG is a planar superhypergraph.

2. *Planar SuperHypergraphs Generalize Planar Hypergraphs.*

   Superhypergraphs introduce additional structures, such as superedges connecting hyperedges or subsets thereof, which are not present in standard hypergraphs. Planar superhypergraphs include all planar hypergraphs and extend them with these complex relationships, while maintaining planarity in their extended bipartite representations. Thus, planar superhypergraphs generalize planar hypergraphs.

3. *Every Outerplanar Hypergraph is an Outerplanar SuperHypergraph.*

   Similarly, for an outerplanar hypergraph $H = (V, E)$, its bipartite representation $B(H)$ is outerplanar. By considering H as a superhypergraph SHG without superedges, $B(SHG) = B(H)$, which is outerplanar. Hence, SHG is an outerplanar superhypergraph.

4. *Outerplanar SuperHypergraphs Generalize Outerplanar Hypergraphs.*

   Outerplanar superhypergraphs encompass all outerplanar hypergraphs and allow for more complex hyperedge structures while preserving outerplanarity in their extended bipartite representations. Therefore, they generalize outerplanar hypergraphs.

□

4.4 Future works: Semi-Directed Planar graph

In the future, I plan to define concepts such as Planar Graphs and Outerplanar Graphs within the context of Semi-Directed Graphs, as well as within frameworks like Fuzzy Semi-Directed Graphs, Neutrosophic Semi-Directed Graphs, Turiyam Neutrosophic Semi-Directed Graphs, and Plithogenic Semi-Directed Graphs.

The definition of a Semi-Directed Graph is provided as follows [111, 112, 171, 471].

Definition 184. Let V be a nonempty set of elements, called vertices or nodes. Let $E_1 \subseteq V \times V$ be a set of unordered pairs of vertices, referred to as undirected edges, and let $E_2 \subseteq V \times V$ be a set of ordered pairs of vertices, referred to as directed edges. A *Semi-Directed Graph* $G = (V, E_1, E_2)$ is a graph where the edge set $E = E_1 \cup E_2$, consisting of both undirected edges from E_1 and directed edges from E_2.

- *Undirected edges*: An edge $(u, v) \in E_1$ implies that there is an undirected connection between vertices u and v, meaning $(u, v) = (v, u)$.
- *Directed edges*: An edge $(a, b) \in E_2$ implies a directed connection from vertex a to vertex b, where $(a, b) \neq (b, a)$.

Although still in the conceptual stage, I have outlined the definition of a Semi-Directed Planar Graph below. By incorporating additional conditions for fuzzy, Neutrosophic, or Plithogenic elements, it is possible to extend this definition to fuzzy, Neutrosophic, or Plithogenic Semi-Directed Planar Graphs.

Definition 185. Let $G = (V, E_1, E_2)$ be a *Semi-Directed Graph* where V is a nonempty set of vertices, $E_1 \subseteq V \times V$ is the set of undirected edges, and $E_2 \subseteq V \times V$ is the set of directed edges. The graph G is called a *Semi-Directed Planar Graph* if there exists a drawing of G on a plane such that:

1. *Planarity condition*: The graph G can be embedded in the plane such that no two edges (whether undirected from E_1 or directed from E_2) intersect except at their endpoints.
2. *Semi-Directed edges*: Both undirected edges $(u, v) \in E_1$ and directed edges $(a, b) \in E_2$ must be drawn in such a way that their connections respect the planarity condition. Specifically:
    - For undirected edges (u, v), the edge is drawn as a straight or curved line between u and v without crossing other edges.
    - For directed edges (a, b), an arrow from a to b is drawn in a planar manner, ensuring no crossings with other edges.

The *Semi-Directed Planar Graph* satisfies the conditions of planarity for both directed and undirected edges simultaneously in a planar representation.

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Data Availability

This paper does not involve any data analysis.

Ethical Approval

This article does not involve any research with human participants or animals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Disclaimer

This study primarily focuses on theoretical aspects, and its application to practical scenarios has not yet been validated. Future research may involve empirical testing and refinement of the proposed methods. The authors have made every effort to ensure that all references cited in this paper are accurate and appropriately attributed. However, unintentional errors or omissions may occur. The authors bear no legal responsibility for inaccuracies in external sources, and readers are encouraged to verify the information provided in the references independently. Furthermore, the interpretations and opinions expressed in this paper are solely those of the authors and do not necessarily reflect the views of any affiliated institutions.

References

[1] Regular interval-valued intuitionistic fuzzy graphs. *Journal of Informatics and Mathematical Sciences*, 9:609–621, 2017.

[2] Perfectly regular and perfectly edge-regular intuitionistic fuzzy graphs. *International Journal of Engineering and Advanced Technology*, 2019.

[3] Ralph Abboud, Ismail Ilkan Ceylan, Martin Grohe, and Thomas Lukasiewicz. The surprising power of graph neural networks with random node initialization. *arXiv preprint arXiv:2010.01179*, 2020.

[4] Mohamed Abdel-Basset, Mohamed El-Hoseny, Abduallah Gamal, and Florentin Smarandache. A novel model for evaluation hospital medical care systems based on plithogenic sets. *Artificial intelligence in medicine*, 100:101710, 2019.

[5] Oludare Isaac Abiodun, Aman Jantan, Abiodun Esther Omolara, Kemi Victoria Dada, Nachaat AbdElatif Mohamed, and Humaira Arshad. State-of-the-art in artificial neural network applications: A survey. *Heliyon*, 4(11), 2018.

[6] Eman A AbuHijleh. Complex hesitant fuzzy graph. *Fuzzy Information and Engineering*, 15(2):149–161, 2023.

[7] Eyal Ackerman, Balázs Keszegh, and Mate Vizer. On the size of planarly connected crossing graphs. *Journal of Graph Algorithms and Applications*, 2017.

[8] Eyal Ackerman and Gábor Tardos. On the maximum number of edges in quasi-planar graphs. *Journal of Combinatorial Theory, Series A*, 114(3):563–571, 2007.

[9] Abhijin Adiga, L Sunil Chandran, and Naveen Sivadasan. Lower bounds for boxicity. *Combinatorica*, 34:631–655, 2014.

[10] Fekadu Tesgera Agama and VN SrinivasaRao Repalle. A study on an extended total fuzzy graph. *Appl. Math*, 16(4):511–518, 2022.

[11] Fekadu Tesgera Agama, VN SrinivasaRao Repalle, and Laxmi Rathour. New results of partially total fuzzy graph. *BMC Research Notes*, 16(1):168, 2023.

[12] Pankaj K. Agarwal, Boris Aronov, János Pach, Richard Pollack, and Micha Sharir. Quasi-planar graphs have a linear number of edges. *Combinatorica*, 17:1–9, 1995.

[13] Alok Aggarwal, C Pandu Rangan, Anil Maheshwari, and Norbert Zeh. External memory algorithms for outerplanar graphs. In *Algorithms and Computation: 10th International Symposium, ISAAC'99 Chennai, India, December 16–18, 1999 Proceedings 10*, pages 307–316. Springer, 1999.

[14] Uzma Ahmad and Tahira Batool. Domination in rough fuzzy digraphs with application. *Soft Computing*, 27(5):2425–2442, 2023.

[15] D Ajay, P Chellamani, et al. Pythagorean neutrosophic fuzzy graphs. *International journal of neutrosophic science*, 11(2):108–114, 2020.

[16] Saeed Akhoondian Amiri, Ali Golshani, Stephan Kreutzer, and Sebastian Siebertz. Vertex disjoint paths in upward planar graphs. In *Computer Science-Theory and Applications: 9th International Computer Science Symposium in Russia, CSR 2014, Moscow, Russia, June 7-11, 2014. Proceedings 9*, pages 52–64. Springer, 2014.

[17] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[18] Muhammad Akram, Uzma Ahmad, Rukhsar, and Faruk Karaaslan. Complex pythagorean fuzzy threshold graphs with application in petroleum replenishment. *Journal of Applied Mathematics and Computing*, 68:2125–2150, 2021.

[19] Muhammad Akram, Muhammad Akram, and Di Cecco. *m- polar fuzzy graphs*. Springer, 2019.

[20] Muhammad Akram, Ayesha Bashir, and Sovan Samanta. Complex pythagorean fuzzy planar graphs. *International Journal of Applied and Computational Mathematics*, 6, 2020.

[21] Muhammad Akram, Ayesha Bashir, and Sovan Samanta. Complex pythagorean fuzzy planar graphs. *International Journal of Applied and Computational Mathematics*, 6:1–27, 2020.

[22] Muhammad Akram, Jawaria Mohsan Dar, and Sumera Naz. Pythagorean dombi fuzzy graphs. *Complex & Intelligent Systems*, 6:29–54, 2020.

[23] Muhammad Akram and Bijan Davvaz. Strong intuitionistic fuzzy graphs. *Filomat*, 26(1):177–196, 2012.

[24] Muhammad Akram and Wieslaw A. Dudek. Regular bipolar fuzzy graphs. *Neural Computing and Applications*, 21:197–205, 2012.

[25] Muhammad Akram and Amna Habib. q-rung picture fuzzy graphs: a creative view on regularity with applications. *Journal of Applied Mathematics and Computing*, 61:235 – 280, 2019.

[26] Muhammad Akram and Ayesha Khan. Complex pythagorean dombi fuzzy graphs for decision making. *Granular Computing*, 6:645–669, 2021.

[27] Muhammad Akram and Anam Luqman. Intuitionistic single-valued neutrosophic hypergraphs. *Opsearch*, 54:799–815, 2017.

[28] Muhammad Akram, Hafsa M Malik, Sundas Shahzadi, and Florentin Smarandache. Neutrosophic soft rough graphs with application. *Axioms*, 7(1):14, 2018.

[29] Muhammad Akram, Sumera Naz, and Bijan Davvaz. Simplified interval-valued pythagorean fuzzy graphs with application. *Complex & intelligent systems*, 5:229–253, 2019.

[30] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar fuzzy planar graphs. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 223–251, 2021.

[31] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.

[32] Muhammad Akram, Sidra Sayed, and Florentin Smarandache. Neutrosophic incidence graphs with application. *Axioms*, 7(3):47, 2018.

[33] Muhammad Akram and Gulfam Shahzadi. *Operations on single-valued neutrosophic graphs*. Infinite Study, 2017.

[34] Muhammad Akram and Gulfam Shahzadi. Decision-making approach based on pythagorean dombi fuzzy soft graphs. *Granular Computing*, 6:671 – 689, 2020.

[35] Muhammad Akram, Sundas Shahzadi, and AB Saeid. Single-valued neutrosophic hypergraphs. *TWMS Journal of Applied and Engineering Mathematics*, 8(1):122–135, 2018.

[36] Muhammad Akram, Ayesha Shareef, and Ahmad N. Al-kenani. Pythagorean fuzzy incidence graphs with application in one-way toll road network. *Granular Computing*, 2024.

[37] Muhammad Akram and KP Shum. *Bipolar neutrosophic planar graphs*. Infinite Study, 2017.

[38] FS Al-Anzi. Efficient cellular automata algorithms for planar graph and vlsi layout homotopic compaction. *International Journal of Computing and Information Sciences*, 1(1):1–17, 2003.

[39] T Ali Al-Hawary. Anti product fuzzy graphs. *International Journal of Applied Mathematics*, 36(6):747, 2023.

[40] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[41] Talal Al-Hawary and Bayan Horani. On product fuzzy graphs. *Annals of fuzzy mathematics and Informatics*, 12(2):279–294, 2016.

[42] Talal Ali Al-Hawary. Density results for perfectly regular and perfectly edge-regular fuzzy graphs. *Journal of Discrete Mathematical Sciences and Cryptography*, pages 1–8, 2022.

[43] Ahmad Al-Jarrah, Mourad Oqla Massa'deh, Abdel Karim Mohamed Baareh, Mohammad A. Aljarrah, Roba Mahmoud Alogla, and Nisrein Jamal sanad Abu-darwish. A study on r-edge regular intuitionistic fuzzy graphs. 2020.

[44] Md Tanvir Alam, Chowdhury Farhan Ahmed, Md Samiullah, and Carson K Leung. Mining frequent patterns from hypergraph databases. In *Pacific-Asia Conference on Knowledge Discovery and Data Mining*, pages 3–15. Springer, 2021.

[45] Marie Albenque, Éric Fusy, and Thomas Lehéricy. Random cubic planar graphs converge to the brownian sphere. *Electronic Journal of Probability*, 28:1–54, 2023.

[46] Jochen Alber, Hans L. Bodlaender, Henning Fernau, Ton Kloks, and Rolf Niedermeier. Fixed parameter algorithms for dominating set and related problems on planar graphs. *Algorithmica*, 33:461–493, 2002.

[47] Saeid Alikhani, Jason I Brown, and Somayeh Jahari. On the domination polynomials of friendship graphs. *Filomat*, 30(1):169–178, 2016.

[48] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[49] Noura Alshehri and Muhammad Akram. Intuitionistic fuzzy planar graphs. *Discrete Dynamics in Nature and Society*, 2014(1):397823, 2014.

[50] Noura Alshehri and andMuhammad Akram. Intuitionistic fuzzy planar graphs. *Discrete Dynamics in Nature and Society*, 2014:1–9, 2014.

[51] Claudi Alsina, Enric Trillas, and Llorenç Valverde. On some logical connectives for fuzzy sets theory. *Journal of Mathematical Analysis and Applications*, 93(1):15–26, 1983.

[52] A Anthoni Amali and J Jesintha Rosline. Oscillating polar fuzzy graph (opfg) and its applications. In *AIP Conference Proceedings*, volume 2852. AIP Publishing, 2023.

[53] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54(7):5255–5281, 2021.

[54] Sk Amanathulla, Biswajit Bera, and Madhumangal Pal. Balanced picture fuzzy graph with application. *Artificial Intelligence Review*, 54:5255 – 5281, 2021.

[55] Patrizio Angelini, Michael A Bekos, Franz J Brandenburg, Giordano Da Lozzo, Giuseppe Di Battista, Walter Didimo, Michael Hoffmann, Giuseppe Liotta, Fabrizio Montecchiani, Ignaz Rutter, et al. Simple k-planar graphs are simple (k+ 1)-quasiplanar. *Journal of Combinatorial Theory, Series B*, 142:1–35, 2020.

[56] Patrizio Angelini, Michael A Bekos, Franz J Brandenburg, Giordano Da Lozzo, Giuseppe Di Battista, Walter Didimo, Giuseppe Liotta, Fabrizio Montecchiani, and Ignaz Rutter. On the relationship between k-planar and k-quasi-planar graphs. In *Graph-Theoretic Concepts in Computer Science: 43rd International Workshop, WG 2017, Eindhoven, The Netherlands, June 21-23, 2017, Revised Selected Papers 43*, pages 59–74. Springer, 2017.

[57] Patrizio Angelini, Michael A Bekos, Michael Kaufmann, Maximilian Pfister, and Torsten Ueckerdt. Beyond-planarity: density results for bipartite graphs. *arXiv preprint arXiv:1712.09855*, 2017.

[58] Patrizio Angelini, Till Bruckdorfer, Giuseppe Di Battista, Michael Kaufmann, Tamara Mchedlidze, Vincenzo Roselli, and Claudio Squarcella. Small universal point sets for k-outerplanar graphs. *Discrete & Computational Geometry*, 60:430–470, 2018.

[59] Patrizio Angelini, Giordano Da Lozzo, Henry Förster, and Thomas Schneck. 2-layer k-planar graphs density, crossing lemma, relationships and pathwidth. *The Computer Journal*, 67(3):1005–1016, 2024.

[60] Patrizio Angelini, Giordano Da Lozzo, Giuseppe Di Battista, Fabrizio Frati, and Maurizio Patrignani. 2-level quasi-planarity or how caterpillars climb (spqr-) trees. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2779–2798. SIAM, 2021.

[61] Renzo Angles and Claudio Gutierrez. Survey of graph database models. *ACM Computing Surveys (CSUR)*, 40(1):1–39, 2008.

[62] Abida Anwar and Faryal Chaudhry. On certain products of complex intuitionistic fuzzy graphs. *Journal of Function Spaces*, 2021.

[63] Lakshmanan S Aparna and A Vijayakumar. *Studies on Some Graph Classes*. PhD thesis, Cochin University of Science and Technology, 2008.

[64] S Archana and PREETHI KUTTIPULACKAL. Line regular fuzzy semigraphs. *Baghdad Science Journal*, 20(1 (SI)):0288–0288, 2023.

[65] Dan Archdeacon, Nora Hartsfield, CHC Little, and Bojan Mohar. Obstruction sets for outer-projective-planar graphs. *Ars combinatoria*, 49:113–128, 1998.

[66] Waqar Arif, Waheed Ahmad Khan, Hossein Rashmanlou, Asghar Khan, and Adnan Muhammad. Multi attribute decision-making and interval-valued picture (s, t)-fuzzy graphs. *Journal of Applied Mathematics and Computing*, 69:2831–2856, 2023.

[67] Takao Asano and Tomio Hirata. Edge-deletion and edge-contraction problems. In *Proceedings of the fourteenth annual ACM symposium on Theory of computing*, pages 245–254, 1982.

[68] Khushbakhat Asif, Muhammad Kamran Jamil, Hanen Karamti, Muhammad Azeem, and Kifayat Ullah. Randić energies for t-spherical fuzzy hamacher graphs and their applications in decision making for business plans. *Computational and Applied Mathematics*, 42:1–32, 2023.

[69] Armen S Asratian, Tristan MJ Denley, and Roland Häggkvist. *Bipartite graphs and their applications*, volume 131. Cambridge university press, 1998.

[70] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[71] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[72] Krassimir T Atanassov and Krassimir T Atanassov. *Intuitionistic fuzzy sets*. Springer, 1999.

[73] Krassimir T. Atanassov, Basil K. Papadopoulos, and Apostolos Syropoulos. An application of the theory of intutionistic fuzzy multigraphs. 2004.

[74] Christopher Auer, Christian Bachmaier, Franz J Brandenburg, Andreas Gleißner, and Kathrin Hanauer. Upward planar graphs and their duals. *Theoretical Computer Science*, 571:36–49, 2015.

[75] Christopher Auer, Christian Bachmaier, Franz J Brandenburg, Andreas Gleißner, Kathrin Hanauer, Daniel Neuwirth, and Josef Reislhuber. Outer 1-planar graphs. *Algorithmica*, 74:1293–1320, 2016.

[76] Maria Axenovich, Carsten Thomassen, Ursula Schade, and Torsten Ueckerdt. Planar ramsey graphs. *arXiv preprint arXiv:1812.00832*, 2018.

[77] Jasine Babu, Areej Khoury, and Ilan Newman. Every property of outerplanar graphs is testable. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2016)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2016.

[78] Christian Bachmaier, Franz J Brandenburg, Kathrin Hanauer, Daniel Neuwirth, and Josef Reislhuber. Nic-planar graphs. *Discrete Applied Mathematics*, 232:23–40, 2017.

[79] Sang Won Bae, Jean-Francois Baffier, Jinhee Chun, Peter Eades, Kord Eickmeyer, Luca Grilli, Seok-Hee Hong, Matias Korman, Fabrizio Montecchiani, Ignaz Rutter, et al. Gap-planar graphs. *Theoretical Computer Science*, 745:36–52, 2018.

[80] Wenhui Bai, Juanjuan Ding, and Chao Zhang. Dual hesitant fuzzy graphs with applications to multi-attribute decision making. *International Journal of Cognitive Computing in Engineering*, 1:18–26, 2020.

[81] David Barnette. Trees in polyhedral graphs. *Canadian Journal of Mathematics*, 18:731–736, 1966.

[82] Marc Barthélemy and Alessandro Flammini. Modeling urban street patterns. *Physical review letters*, 100(13):138702, 2008.

[83] Mehdi Behzad, Gary Chartrand, and John K Cooper Jr. The colour numbers of complete graphs. *Journal of the London Mathematical Society*, 1(1):226–228, 1967.

[84] Michael A Bekos. k-planar graphs. In *Beyond Planar Graphs: Communications of NII Shonan Meetings*, pages 109–130. Springer, 2020.

[85] Michael A Bekos, Sabine Cornelsen, Luca Grilli, Seok-Hee Hong, and Michael Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. *Algorithmica*, 79:401–427, 2017.

[86] Michael A Bekos and Luca Grilli. Fan-planar graphs. In *Beyond Planar Graphs: Communications of NII Shonan Meetings*, pages 131–148. Springer, 2020.

[87] Michael A Bekos, Michael Kaufmann, and Chrysanthi N Raftopoulou. On the density of non-simple 3-planar graphs. In *International Symposium on Graph Drawing and Network Visualization*, pages 344–356. Springer, 2016.

[88] Michael A Bekos, Michael Kaufmann, and Chrysanthi N Raftopoulou. On optimal 2-and 3-planar graphs. *arXiv preprint arXiv:1703.06526*, 2017.

[89] Michael A Bekos and Chrysanthi N Raftopoulou. Circle-representations of simple 4-regular planar graphs. In *International Symposium on Graph Drawing*, pages 138–149. Springer, 2012.

[90] Nuel D Belnap Jr. A useful four-valued logic. In *Modern uses of multiple-valued logic*, pages 5–37. Springer, 1977.

[91] Boaz Ben-Moshe, Amit Dvir, Michael Segal, and Arie Tamir. Centdian computation in cactus graphs. *Journal of Graph Algorithms and Applications*, 16(2):199–224, 2012.

[92] Biswajit Bera, Sk Amanathulla, and Sanat Kumar Mahato. A comprehensive study of picture fuzzy planar graphs with real-world applications. *Journal of uncertain Systems*, 16(04):2350009, 2023.

[93] André Berger, Artur Czumaj, Michelangelo Grigni, and Hairong Zhao. Approximation schemes for minimum 2-connected spanning subgraphs in weighted planar graphs. In *Algorithms–ESA 2005: 13th Annual European Symposium, Palma de Mallorca, Spain, October 3-6, 2005. Proceedings 13*, pages 472–483. Springer, 2005.

[94] Ivona Bezáková and Adam J Friedlander. Counting minimum (s, t)-cuts in weighted planar graphs in polynomial time. In *International Symposium on Mathematical Foundations of Computer Science*, pages 126–137. Springer, 2010.

[95] T Bharathi, S Felixia, and S Leo. Intuitionistic felicitous fuzzy graphs.

[96] Anushree Bhattacharya and Madhumangal Pal. A fuzzy graph theory approach to the facility location problem: A case study in the indian banking system. *Mathematics*, 11(13):2992, 2023.

[97] Therese Biedl. On triangulating k-outerplanar graphs. *Discrete Applied Mathematics*, 181:275–279, 2015.

[98] Therese Biedl. Drawing outer-1-planar graphs revisited. *arXiv preprint arXiv:2009.07106*, 2020.

[99] Therese Biedl and Debajyoti Mondal. Improved outerplanarity bounds for planar graphs. *arXiv preprint arXiv:2407.04282*, 2024.

[100] Carla Binucci, Aaron Büngener, Giuseppe Di Battista, Walter Didimo, Vida Dujmović, Seok-Hee Hong, Michael Kaufmann, Giuseppe Liotta, Pat Morin, and Alessandra Tappini. Min-k-planar drawings of graphs. In *International Symposium on Graph Drawing and Network Visualization*, pages 39–52. Springer, 2023.

[101] Carla Binucci, Emilio Di Giacomo, Walter Didimo, Fabrizio Montecchiani, Maurizio Patrignani, and Ioannis G Tollis. Fan-planar graphs: combinatorial properties and complexity results. In *Graph Drawing: 22nd International Symposium, GD 2014, Würzburg, Germany, September 24-26, 2014, Revised Selected Papers 22*, pages 186–197. Springer, 2014.

[102] Carla Binucci, Emilio Di Giacomo, Walter Didimo, Fabrizio Montecchiani, Maurizio Patrignani, and Ioannis G Tollis. Properties and complexity of fan-planarity. *arXiv preprint arXiv:1406.5299*, 2014.

[103] Siddhartha Sankar Biswas, Bashir Alam, and MN Doja. A generalized real time multigraphs for communication networks: An intuitionistic fuzzy theoretical model. In *17th International Conference on IFS, Sofia, Bulgaria, Proceedings published in Notes on Intuitionistic Fuzzy Sets (Bulgarian Journal) Vol*, volume 19, 2013.

[104] R Bodendiek and R Henn. The outerthickness & outercoarseness of graphs i. the complete graph & the n-cube. *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, page 297, 1990.

[105] Manuel Bodirsky, Clemens Gröpl, and Mihyun Kang. Sampling unlabeled biconnected planar graphs. In *Algorithms and Computation: 16th International Symposium, ISAAC 2005, Sanya, Hainan, China, December 19-21, 2005. Proceedings 16*, pages 593–603. Springer, 2005.

[106] Manuel Bodirsky and Mihyun Kang. Generating outerplanar graphs uniformly at random. *Combinatorics, Probability and Computing*, 15(3):333–343, 2006.

[107] Hans L Bodlaender. A tourist guide through treewidth. *Acta cybernetica*, 11(1-2):1–21, 1993.

[108] Hans L Bodlaender. Treewidth: Algorithmic techniques and results. In *International Symposium on Mathematical Foundations of Computer Science*, pages 19–36. Springer, 1997.

[109] Hans L Bodlaender. A partial k-arboretum of graphs with bounded treewidth. *Theoretical computer science*, 209(1-2):1–45, 1998.

[110] Hans L Bodlaender and Arie MCA Koster. Treewidth computations ii. lower bounds. *Information and Computation*, 209(7):1103–1119, 2011.

[111] Anthony Bonato, Dejan Delic, and Changping Wang. Universal random semi-directed graphs. *Proceedings of ROGICS'08*, pages 138–148, 2008.

[112] Anthony Bonato, DEJAN DELIĆ, and Changping Wang. The structure and automorphisms of semi-directed graphs. *Journal of Multiple-Valued Logic & Soft Computing*, 27, 2016.

[113] J Adrian Bondy and László Lovász. Lengths of cycles in halin graphs. *Journal of graph theory*, 9(3):397–410, 1985.

[114] John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. *Graph theory with applications*, volume 290. Macmillan London, 1976.

[115] Édouard Bonnet. Treewidth inapproximability and tight eth lower bound. *arXiv preprint arXiv:2406.11628*, 2024.

[116] Flavia Bonomo-Braberman, Guillermo Durán, Martín D Safe, and Annegret K Wagler. On some graph classes related to perfect graphs: A survey. *Discrete Applied Mathematics*, 281:42–60, 2020.

[117] Christian Borgs, Jennifer Chayes, László Lovász, Vera T Sós, and Katalin Vesztergombi. Counting graph homomorphisms. In *Topics in Discrete Mathematics: Dedicated to Jarik Nešetřil on the Occasion of his 60th Birthday*, pages 315–371. Springer, 2006.

[118] Glencora Borradaile, Philip N Klein, Shay Mozes, Yahav Nussbaum, and Christian Wulff-Nilsen. Multiple-source multiple-sink maximum flow in directed planar graphs in near-linear time. *SIAM Journal on Computing*, 46(4):1280–1303, 2017.

[119] Rajab Ali Borzooei and R. Almallah. Inverse fuzzy multigraphs and planarity with application in decision-making. *Soft Computing*, 26:1531 – 1539, 2021.

[120] Rajab Ali Borzooei, R Almallah, Young Bae Jun, and H Ghaznavi. Inverse fuzzy graphs with applications. *New Mathematics and Natural Computation*, 16(02):397–418, 2020.

[121] André Bouchet. Circle graph obstructions. *Journal of Combinatorial Theory, Series B*, 60(1):107–144, 1994.

[122] Franz J Brandenburg. A simple quasi-planar drawing of k10. In *Graph Drawing*, volume 9801, pages 603–604, 2016.

[123] Franz J Brandenburg. On fan-crossing and fan-crossing free graphs. *Information Processing Letters*, 138:67–71, 2018.

[124] Franz J Brandenburg. On fan-crossing graphs. *Theoretical Computer Science*, 841:39–49, 2020.

[125] Franz J Brandenburg, Walter Didimo, William S Evans, Philipp Kindermann, Giuseppe Liotta, and Fabrizio Montecchiani. Recognizing and drawing ic-planar graphs. *Theoretical Computer Science*, 636:1–16, 2016.

[126] Andreas Brandstädt, Van Bang Le, and Jeremy P Spinrad. *Graph classes: a survey*. SIAM, 1999.

[127] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[128] Said Broumi, Assia Bakali, and Ayoub Bahnasse. Neutrosophic sets: An overview. *Infinite Study*, 2018.

[129] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and V Venkateswara Rao. Interval complex neutrosophic graph of type. *Collected Papers. Volume XIV: Neutrosophics and other topics*, page 289, 2022.

[130] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[131] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.

[132] Sergio Cabello. Subquadratic algorithms for the diameter and the sum of pairwise distances in planar graphs. *ACM Transactions on Algorithms (TALG)*, 15:1 – 38, 2017.

[133] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, Shenda Hong, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *IJCAI*, pages 1923–1929, 2022.

[134] Gruia Călinescu and Cristina G Fernandes. Maximum planar subgraph. In *Handbook of Approximation Algorithms and Metaheuristics*, pages 171–186. Chapman and Hall/CRC, 2018.

[135] Renato M Capocelli and Aldo De Luca. Fuzzy sets and decision theory. *Information and control*, 23(5):446–473, 1973.

[136] Alessio Cardillo, Salvatore Scellato, Vito Latora, and Sergio Porta. Structural properties of planar graphs of urban street patterns. *Physical Review Statistical, Nonlinear, and Soft Matter Physics*, 73(6):066107, 2006.

[137] Michael Cary. Perfectly regular and perfectly edge-regular fuzzy graphs. *Annals of Pure and Applied Mathematics*, 16(2):461–469, 2018.

[138] Chun-Hsiang Chan and Hsu-Chun Yen. On contact representations of directed planar graphs. In *Computing and Combinatorics: 24th International Conference, COCOON 2018, Qing Dao, China, July 2-4, 2018, Proceedings 24*, pages 218–229. Springer, 2018.

[139] L Sunil Chandran and Naveen Sivadasan. Boxicity and treewidth. *Journal of Combinatorial Theory, Series B*, 97(5):733–744, 2007.

[140] Yi-Jun Chang and Hsu-Chun Yen. Area-universal drawings of biconnected outerplane graphs. *Information Processing Letters*, 118:1–5, 2017.

[141] Steven Chaplick, Myroslav Kryven, Giuseppe Liotta, Andre Löffler, and Alexander Wolff. Beyond outerplanarity. In *International Symposium on Graph Drawing and Network Visualization*, pages 546–559. Springer, 2017.

[142] Gary Chartrand. *Introductory graph theory*. Courier Corporation, 2012.

[143] Chandra Chekuri, Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Embedding k-outerplanar graphs into l1. *SIAM Journal on Discrete Mathematics*, 20(1):119–136, 2006.

[144] P. Chellamani and D. Ajay. Pythagorean neutrosophic dombi fuzzy graphs with an application to mcdm. 2021.

[145] P. Chellamani and D. Ajay. Spherical fuzzy labelling graphs. *Journal of Physics: Conference Series*, 1964, 2021.

[146] P Chellamani, D Ajay, Mohammed M Al-Shamiri, and Rashad Ismail. *Pythagorean Neutrosophic Planar Graphs with an Application in Decision-Making*. Infinite Study, 2023.

[147] P Chellamani, D Ajay, Said Broumi, and T Antony Alphonse Ligori. An approach to decision-making via picture fuzzy soft graphs. *Granular computing*, pages 1–22, 2021.

[148] Chiuyuan Chen, Chin-Chen Chang, and Gerard J Chang. Proper interval graphs and the guard problem. *Discrete Mathematics*, 170(1-3):223–230, 1997.

[149] Min Chen, André Raspaud, Weifan Wang, and Weiqiang Yu. Partitioning planar graph of girth 5 into two forests with maximum degree 4. *Czechoslovak Mathematical Journal*, pages 1–12, 2024.

[150] Tinghuai Chen and Tinghuai Chen. Four-valued logic and its applications. *Fault diagnosis and fault tolerance: a systematic approach to special topics*, pages 1–64, 1992.

[151] Zhihua Chen, Waheed Ahmad Khan, and Aysha Khan. Concepts of picture fuzzy line graphs and their applications in data analysis. *Symmetry*, 15(5):1018, 2023.

[152] Lei Cheng, Liang Deng, and Martin DF Wong. Floorplanning for 3-d vlsi design. In *Proceedings of the 2005 Asia and South Pacific Design Automation Conference*, pages 405–411, 2005.

[153] Otfried Cheong, Sariel Har-Peled, Heuna Kim, and Hyo-Sil Kim. On the number of edges of fan-crossing free graphs. *Algorithmica*, 73:673–695, 2015.

[154] Paul Chew. There is a planar graph almost as good as the complete graph. In *Proceedings of the second annual symposium on Computational geometry*, pages 169–177, 1986.

[155] Eli Chien, Chao Pan, Jianhao Peng, and Olgica Milenkovic. You are allset: A multiset function framework for hypergraph neural networks. *arXiv preprint arXiv:2106.13264*, 2021.

[156] Markus Chimani, Torben Donzelmann, Nick Kloster, Melissa Koch, Jan-Jakob Völlering, and Mirko H Wagner. Crossing numbers of beyond planar graphs re-revisited: A framework approach. *arXiv preprint arXiv:2407.05057*, 2024.

[157] Markus Chimani, Carsten Gutwenger, Michael Jünger, Gunnar W Klau, Karsten Klein, and Petra Mutzel. The open graph drawing framework (ogdf). *Handbook of graph drawing and visualization*, 2011:543–569, 2013.

[158] Markus Chimani, Petr Hliněný, and Petra Mutzel. Vertex insertion approximates the crossing number of apex graphs. *European Journal of Combinatorics*, 33(3):326–335, 2012.

[159] Fan RK Chung. Universal graphs and induced-universal graphs. *Journal of Graph Theory*, 14(4):443–454, 1990.

[160] Fan RK Chung, Frank Thomson Leighton, and Arnold L Rosenberg. Embedding graphs in books: a layout problem with applications to vlsi design. *SIAM Journal on Algebraic Discrete Methods*, 8(1):33–58, 1987.

[161] Julia Chuzhoy, Yury Makarychev, and Anastasios Sidiropoulos. On graph crossing number and edge planarization. In *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete algorithms*, pages 1050–1069. SIAM, 2011.

[162] Vašek Chvátal and Najiba Sbihi. Bull-free berge graphs are perfect. *Graphs and Combinatorics*, 3(1):127–139, 1987.

[163] Iwona Cieslik, Marcin Kozik, and Piotr Micek. On-line coloring of is-free graphs and co-planar graphs. *Proceedings of the Annual Discrete Mathematics and Theoretical Computer Science AF, Nancy, France*, pages 61–68, 2006.

[164] Brent N Clark, Charles J Colbourn, and David S Johnson. Unit disk graphs. *Discrete mathematics*, 86(1-3):165–177, 1990.

[165] Manuel J Cobo, Antonio Gabriel López-Herrera, Enrique Herrera-Viedma, and Francisco Herrera. An approach for detecting, quantifying, and visualizing the evolution of a research field: A practical application to the fuzzy sets theory field. *Journal of informetrics*, 5(1):146–166, 2011.

[166] Charles J. Colbourn and Kellogg S. Booth. Linear time automorphism algorithms for trees, interval graphs, and planar graphs. *SIAM J. Comput.*, 10:203–225, 1981.

[167] Grard Cornujols, Denis Naddef, and William R Pulleyblank. Halin graphs and the travelling salesman problem. *Mathematical programming*, 26(3):287–294, 1983.

[168] Ido Dagan, Martin Charles Golumbic, and Ron Yair Pinter. Trapezoid graphs and their coloring. *Discrete Applied Mathematics*, 21(1):35–46, 1988.

[169] Christine Dahn, Nils M Kriege, and Petra Mutzel. A fixed-parameter algorithm for the max-cut problem on embedded 1-planar graphs. In *International Workshop on Combinatorial Algorithms*, pages 141–152. Springer, 2018.

[170] Christine Dahn, Nils M Kriege, Petra Mutzel, and Julian Schilling. Fixed-parameter algorithms for the weighted max-cut problem on embedded 1-planar graphs. *Theoretical Computer Science*, 852:172–184, 2021.

[171] Rakhi Das, Laxminarayan Sahoo, Sovan Samanta, Vladimir Simic, and Tapan Senapati. Identifying the shortest path of a semidirected graph and its application. *Mathematics*, 10(24):4807, 2022.

[172] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[173] Sankar Das, Soumitra Poulik, and Ganesh Ghorai. Picture fuzzy ϕ-tolerance competition graphs with its application. *Journal of Ambient Intelligence and Humanized Computing*, 15(1):547–559, 2024.

[174] Suman Das, Rakhal Das, and Surapati Pramanik. Single valued pentapartitioned neutrosophic graphs. *Neutrosophic Sets and Systems*, 50(1):225–238, 2022.

[175] Bijan Davvaz, Naeem Jan, Tahir Mahmood, and Kifayat Ullah. Intuitionistic fuzzy graphs of n th type with applications. *Journal of Intelligent & Fuzzy Systems*, 36(4):3923–3932, 2019.

[176] Supriya Kumar De, Ranjit Biswas, and Akhil Ranjan Roy. Some operations on intuitionistic fuzzy sets. *Fuzzy sets and Systems*, 114(3):477–484, 2000.

[177] Hubert De Fraysseix, János Pach, and Richard Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.

[178] Hubert de Fraysseix, János Pach, and Richard Pollack. How to draw a planar graph on a grid. *Combinatorica*, 10:41–51, 1990.

[179] Aldo De Luca and Settimo Termini. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory. In *Readings in fuzzy sets for intelligent systems*, pages 197–202. Elsevier, 1993.

[180] Marisa Debowsky. *Results on planar hypergraphs and on cycle decompositions*. PhD thesis, University of Vermont, 2002.

[181] Hooman R Dehkordi and Graham Farr. Non-separating planar graphs. *arXiv preprint arXiv:1907.09817*, 2019.

[182] Narsingh Deo. *Graph theory with applications to engineering and computer science*. Courier Dover Publications, 2016.

[183] Rémi Desgranges and Kolja Knauer. A correction of a characterization of planar partial cubes. *Discrete Mathematics*, 340(6):1151–1153, 2017.

[184] N Deva and A Felix. Bipolar intuitionistic anti fuzzy graphs. In *AIP Conference Proceedings*, volume 2385. AIP Publishing, 2022.

[185] N Deva and A. Felix. Bipolar intuitionistic fuzzy competition graphs. *Journal of Physics: Conference Series*, 2267, 2022.

[186] PM Dhanya and PB Ramkumar. Text analysis using morphological operations on a neutrosophic text hypergraph. *Neutrosophic Sets and Systems*, 61:337–364, 2023.

[187] Giuseppe Di Battista, Roberto Tamassia, and Luca Vismara. On-line convex planarity testing. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 242–255. Springer, 1994.

[188] Emilio Di Giacomo, Walter Didimo, Giuseppe Liotta, Henk Meijer, and Stephen K Wismath. Planar and quasi-planar simultaneous geometric embedding. *The Computer Journal*, 58(11):3126–3140, 2015.

[189] Emilio Di Giacomo, Walter Didimo, Giuseppe Liotta, and Fabrizio Montecchiani. h-quasi planar drawings of bounded treewidth graphs in linear area. In *International Workshop on Graph-Theoretic Concepts in Computer Science*, pages 91–102. Springer, 2012.

[190] Walter Didimo. Density of straight-line 1-planar graph drawings. *Information Processing Letters*, 113(7):236–240, 2013.

[191] Walter Didimo, Peter Eades, and Giuseppe Liotta. Drawing graphs with right angle crossings. *Theoretical Computer Science*, 412(39):5156–5166, 2011.

[192] Walter Didimo and Giuseppe Liotta. The crossing-angle resolution in graph drawing. *Thirty Essays on Geometric Graph Theory*, pages 167–184, 2013.

[193] Walter Didimo, Giuseppe Liotta, and Fabrizio Montecchiani. A survey on graph drawing beyond planarity. *ACM Computing Surveys (CSUR)*, 52(1):1–37, 2019.

[194] Reinhard Diestel. Graduate texts in mathematics: Graph theory.

[195] Reinhard Diestel. Graph theory 3rd ed. *Graduate texts in mathematics*, 173(33):12, 2005.

[196] Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.

[197] Radoslav Dimitrov, Zeyang Zhao, Ralph Abboud, and Ismail Ceylan. Plane: representation learning over planar graphs. *Advances in Neural Information Processing Systems*, 36:16028–16054, 2023.

[198] T Dinesh. Fuzzy incidence graph structures. *Adv. Fuzzy Math.(AFM)*, 15(1):21–30, 2020.

[199] Guoli Ding and Stan Dziobiak. Excluded-minor characterization of apex-outerplanar graphs. *Graphs and Combinatorics*, 32:583 – 627, 2015.

[200] Guoli Ding and Stan Dziobiak. Excluded-minor characterization of apex-outerplanar graphs. *Graphs and Combinatorics*, 32(2):583–627, 2016.

[201] Guoli Ding, Joshua Fallon, and Emily Marshall. On almost-planar graphs. *arXiv preprint arXiv:1603.02310*, 2016.

[202] Guoli Ding, Bogdan Oporowski, Robin Thomas, and Dirk Vertigan. Large non-planar graphs and an application to crossing-critical graphs. *Journal of Combinatorial Theory, Series B*, 101(2):111–121, 2011.

[203] GABRIEL ANDREW Dirac. Homomorphism theorems for graphs. *Mathematische Annalen*, 153(1):69–80, 1964.

[204] David P Dobkin, Steven J Friedman, and Kenneth J Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete & Computational Geometry*, 5:399–407, 1990.

[205] Yihe Dong, Will Sawin, and Yoshua Bengio. Hnhn: Hypergraph networks with hyperedge neurons. *arXiv preprint arXiv:2006.12278*, 2020.

[206] Didier Dubois and Henri Prade. *Fundamentals of fuzzy sets*, volume 7. Springer Science & Business Media, 2012.

[207] Andrew L Dulmage and Nathan S Mendelsohn. Coverings of bipartite graphs. *Canadian Journal of Mathematics*, 10:517–534, 1958.

[208] Zdeněk Dvořák et al. On planar mixed hypergraphs. *the electronic journal of combinatorics*, pages R35–R35, 2001.

[209] Stan Dziobiak and Gioli Ding. Obstructions of apex classes of graphs. *Unpublished results (see http://msdiscretemath. org/2013/dziobiak. pdf)*, 5, 2013.

[210] Peter Eades, Brendan D McKay, and Nicholas C Wormald. On an edge crossing problem. In *Proc. 9th Australian Computer Science Conference*, volume 327, page 334, 1986.

[211] PA Ejegwa, SO Akowe, PM Otene, and JM Ikyule. An overview on intuitionistic fuzzy sets. *Int. J. Sci. Technol. Res*, 3(3):142–145, 2014.

[212] Nancy El-Hefenawy, Mohamed A Metwally, Zenat M Ahmed, and Ibrahim M El-Henawy. A review on the applications of neutrosophic sets. *Journal of Computational and Theoretical Nanoscience*, 13(1):936–944, 2016.

[213] David Elliott. *Computing Obstruction Sets for Almost Outerplanar Graphs*. PhD thesis, ResearchSpace@ Auckland, 2015.

[214] Julian Enoch, Kyle Fox, Dor Mesica, and Shay Mozes. Faster algorithm for maximum flow in directed planar graphs with vertex capacities. *arXiv preprint arXiv:2101.11300*, 2021.

[215] Hikoe Enomoto, Tomoki Nakamigawa, and Katsuhiro Ota. On the pagenumber of complete bipartite graphs. *journal of combinatorial theory, Series B*, 71(1):111–120, 1997.

[216] William Evans, Michael Kaufmann, William Lenhart, Tamara Mchedlidze, and Stephen Wismath. Bar 1-visibility graphs and their relation to other nearly planar graphs. *Journal of Graph Algorithms and Applications*, 18(5):721–739, 2014.

[217] Ronald Fagin. Degrees of acyclicity for hypergraphs and relational database schemes. *Journal of the ACM (JACM)*, 30(3):514–550, 1983.

[218] Wenfei Fan, Jianzhong Li, Shuai Ma, Hongzhi Wang, and Yinghui Wu. Graph homomorphism revisited for graph matching. *Proceedings of the VLDB Endowment (PVLDB)*, 3(1):1161–1172, 2010.

[219] Wenqi Fan, Yao Ma, Qing Li, Yuan He, Eric Zhao, Jiliang Tang, and Dawei Yin. Graph neural networks for social recommendation. In *The world wide web conference*, pages 417–426, 2019.

[220] Olivier Faugeras. *Three-dimensional computer vision: a geometric viewpoint*. MIT press, 1993.

[221] Song Feng, Emily Heath, Brett Jefferson, Cliff Joslyn, Henry Kvinge, Hugh D Mitchell, Brenda Praggastis, Amie J Eisfeld, Amy C Sims, Larissa B Thackray, et al. Hypergraph models of biological networks to identify genes critical to pathogenic viral response. *BMC bioinformatics*, 22(1):287, 2021.

[222] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.

[223] Zhou Feng, Bo Yao, and Chung-Kuan Cheng. Floorplan representation in vlsi. In *Handbook of Data Structures and Applications*, pages 833–855. Chapman and Hall/CRC, 2018.

[224] Oksana Firman, Grzegorz Gutowski, Myroslav Kryven, Yuto Okada, and Alexander Wolff. Bounding the treewidth of outer k-planar graphs via triangulations. *arXiv preprint arXiv:2408.04264*, 2024.

[225] AD Forbes and Terry S Griggs. Archimedean graph designs. *Discrete Mathematics*, 313(11):1138–1149, 2013.

[226] Henry Förster, Michael Kaufmann, Laura Merker, Sergey Pupyrev, and Chrysanthi Raftopoulou. Linear layouts of bipartite planar graphs. In *Algorithms and Data Structures Symposium*, pages 444–459. Springer, 2023.

[227] Henry Förster, Michael Kaufmann, and Chrysanthi N Raftopoulou. Recognizing and embedding simple optimal 2-planar graphs. In *International Symposium on Graph Drawing and Network Visualization*, pages 87–100. Springer, 2021.

[228] David A Forsyth and Jean Ponce. *Computer vision: a modern approach*. prentice hall professional technical reference, 2002.

[229] Ulrich Fößmeier, Goos Kant, and Michael Kaufmann. 2-visibility drawings of planar graphs. In *Graph Drawing: Symposium on Graph Drawing, GD'96 Berkeley, California, USA, September 18–20, 1996 Proceedings 4*, pages 155–168. Springer, 1997.

[230] J Joseph Fowler. Strongly-connected outerplanar graphs with proper touching triangle representations. In *International Symposium on Graph Drawing*, pages 155–160. Springer, 2013.

[231] Jacob Fox, János Pach, and Andrew Suk. The number of edges in k-quasi-planar graphs. *SIAM Journal on Discrete Mathematics*, 27(1):550–561, 2013.

[232] Mathew C Francis, Jan Kratochvíl, and Tomáš Vyskočil. Segment representation of a subclass of co-planar graphs. *Discrete Mathematics*, 312(10):1815–1818, 2012.

[233] Max Franz, Christian T Lopes, Gerardo Huck, Yue Dong, Onur Sumer, and Gary D Bader. Cytoscape. js: a graph theory library for visualisation and analysis. *Bioinformatics*, 32(2):309–311, 2016.

[234] Takaaki Fujita. General, general weak, anti, balanced, and semi-neutrosophic graph.

[235] Takaaki Fujita. A review of the hierarchy of plithogenic, neutrosophic, and fuzzy graphs: Survey and applications. *ResearchGate(Preprint)*, 2024.

[236] Takaaki Fujita. Short note of supertree-width and n-superhypertree-width. *Neutrosophic Sets and Systems*, 77:54–78, 2024.

[237] Takaaki Fujita. Superhypertree-width and neutrosophictree-width. *preprint(researchgate)*, 2024.

[238] Takaaki Fujita. Various properties of various ultrafilters, various graph width parameters, and various connectivity systems. *arXiv preprint arXiv:2408.02299*, 2024.

[239] Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.

[240] Takaaki Fujita and Florentin Smarandache. A concise study of some superhypergraph classes. *Neutrosophic Sets and Systems*, 77:548–593, 2024.

[241] Takaaki Fujita and Florentin Smarandache. Study for general plithogenic soft expert graphs. *Plithogenic Logic and Computation*, 2:107–121, 2024.

[242] Takaaki Fujita and Florentin Smarandache. Uncertain automata and uncertain graph grammar. *Neutrosophic Sets and Systems*, 74:128–191, 2024.

[243] Delbert Fulkerson and Oliver Gross. Incidence matrices and interval graphs. *Pacific journal of mathematics*, 15(3):835–855, 1965.

[244] GA Ganati, VNS Rao Repalle, MA Ashebo, and M Amini. Turiyam graphs and its applications. *Information Sciences Letters*, 12(6):2423–2434, 2023.

[245] Gamachu Adugna Ganati, VN Srinivasa Rao Repalle, and Mamo Abebe Ashebo. Social network analysis by turiyam graphs. *BMC Research Notes*, 16(1):170, 2023.

[246] Gamachu Adugna Ganati, VN Srinivasa Rao Repalle, and Mamo Abebe Ashebo. Relations in the context of turiyam sets. *BMC Research Notes*, 16(1):49, 2023.

[247] A. Nagoor Gani and S. Shajitha Begum. Perfect intuitionistic fuzzy graphs. *Bulletin of Pure & Applied Sciences-Mathematics and Statistics*, pages 145–152, 2011.

[248] Robert Ganian, Fabian Klute, and Sebastian Ordyniak. On structural parameterizations of the bounded-degree vertex deletion problem. *Algorithmica*, 83:297–336, 2021.

[249] Yue Gao, Yifan Feng, Shuyi Ji, and Rongrong Ji. Hgnn+: General hypergraph neural networks. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(3):3181–3199, 2022.

[250] H Garrett. Neutrosophic 1-failed superhyperforcing in the superhyperfunction to use neutrosophic superhypergraphs on cancer's neutrosophic recognition and beyond. *J Math Techniques Comput Math*, 2(6):221–307, 2023.

[251] Fanica Gavril. Algorithms on circular-arc graphs. *Networks*, 4(4):357–369, 1974.

[252] S. P. Geetha and S. Praveena. Distinct concept of edge regular intuitionistic fuzzy m-polargraphs. *ScieXplore: International Journal of Research in Science*, 2018.

[253] Ellen Gethner, Bopanna Kallichanda, Alexander S Mentis, Sarah Braudrick, Sumeet Chawla, Andrew Clune, Rachel Drummond, Panagiota Evans, William Roche, and Nao Takano. How false is kempe's proof of the four color theorem? part ii. *Involve*, 2(3):249–266, 2009.

[254] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[255] Kamran Gholamizadeh, Esmaeil Zarei, Mohsen Omidvar, and Mohammad Yazdi. Fuzzy sets theory and human reliability: Review, applications, and contributions. *Linguistic methods under fuzzy information in system safety and reliability analysis*, pages 91–137, 2022.

[256] Ganesh Ghorai and Madhumangal Pal. Faces and dual of m-polar fuzzy planar graphs. *J. Intell. Fuzzy Syst.*, 31:2043–2049, 2016.

[257] Ganesh Ghorai and Madhumangal Pal. Faces and dual of m-polar fuzzy planar graphs. *Journal of Intelligent & Fuzzy Systems*, 31(3):2043–2049, 2016.

[258] Ganesh Ghorai and Madhumangal Pal. A study on m-polar fuzzy planar graphs. *International Journal of Computing Science and Mathematics*, 7(3):283–292, 2016.

[259] Omer Giménez and Marc Noy. The number of planar graphs and properties of random planar graphs. *Discrete Mathematics & Theoretical Computer Science*, (Proceedings), 2005.

[260] Puspendu Giri, Somnath Paul, and Bijoy Krishna Debnath. A fuzzy graph theory and matrix approach (fuzzy gtma) to select the best renewable energy alternative in india. *Applied Energy*, 358:122582, 2024.

[261] Wayne Goddard and Michael A Henning. Domination in planar graphs with small diameter. *Journal of Graph Theory*, 40(1):1–25, 2002.

[262] A Goldner and Frank Harary. Note on a smallest nonhamiltonian maximal planar graph. *Bull. Malaysian Math. Soc*, 6(1):41–42, 1975.

[263] Petr A Golovach, Dieter Kratsch, Daniël Paulusma, and Anthony Stewart. A linear kernel for finding square roots of almost planar graphs. *Theoretical Computer Science*, 689:36–47, 2017.

[264] S Gomathy, D Nagarajan, S Broumi, and M Lathamaheswari. *Plithogenic sets and their application in decision making*. Infinite Study, 2020.

[265] Zengtai Gong and Junhu Wang. Hesitant fuzzy graphs, hesitant fuzzy hypergraphs and fuzzy graph decisions. *Journal of Intelligent & Fuzzy Systems*, 40(1):865–875, 2021.

[266] Sathyanarayanan Gopalakrishnan, Supriya Sridharan, Soumya Ranjan Nayak, Janmenjoy Nayak, and Swaminathan Venkataraman. Central hubs prediction for bio networks by directed hypergraph-ga with validation to covid-19 ppi. *Pattern Recognition Letters*, 153:246–253, 2022.

[267] Georg Gottlob, Nicola Leone, and Francesco Scarcello. Hypertree decompositions: A survey. In *Mathematical Foundations of Computer Science 2001: 26th International Symposium, MFCS 2001 Mariánské Lázne, Czech Republic, August 27–31, 2001 Proceedings 26*, pages 37–57. Springer, 2001.

[268] Alexander Grigoriev, Athanassios Koutsonas, and Dimitrios M Thilikos. Nearly planar graphs and {\lambda}-flat graphs. *arXiv preprint arXiv:1311.0137*, 2013.

[269] Jonathan Gross. Genus distributions of cubic outerplanar graphs. *Journal of Graph Algorithms and Applications*, 15(2):295–316, 2011.

[270] Jonathan L Gross. Embeddings of cubic halin graphs: Genus distributions. *ARS Math. Contemp.*, 6(1):37–56, 2013.

[271] Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.

[272] Branko Grünbaum. Some analogues of eberhard's theorem on convex polytopes. *Israel journal of mathematics*, 6:398–411, 1968.

[273] D. J. Guan and Xuding Zhu. Game chromatic number of outerplanar graphs. *J. Graph Theory*, 30:67–70, 1999.

[274] Bradley S Gubser. A characterization of almost-planar graphs. *Combinatorics, Probability and Computing*, 5(3):227–245, 1996.

[275] Abhishek Guleria and Rakesh Kumar Bajaj. T-spherical fuzzy graphs: operations and applications in various selection processes. *Arabian Journal for Science and Engineering*, 45(3):2177–2193, 2020.

[276] Muhammad Gulistan, Naveed Yaqoob, Zunaira Rashid, Florentin Smarandache, and Hafiz Abdul Wahab. A study on neutrosophic cubic graphs with real life applications in industries. *Symmetry*, 10(6):203, 2018.

[277] Kevin Gurney. *An introduction to neural networks*. CRC press, 2018.

[278] Frank O Hadlock. A shortest path algorithm for grid graphs. *Networks*, 7(4):323–334, 1977.

[279] Rudolf Halin. A theorem on n-connected graphs. *Journal of Combinatorial Theory*, 7(2):150–154, 1969.

[280] Mohammad Hamidi, Florentin Smarandache, and Elham Davneshvar. Spectrum of superhypergraphs via flows. *Journal of Mathematics*, 2022(1):9158912, 2022.

[281] Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.

[282] Ming Han, Tao Wang, Jianglin Wu, Huan Zhou, and Xuding Zhu. Weak degeneracy of planar graphs and locally planar graphs. *arXiv preprint arXiv:2303.07901*, 2023.

[283] Frank Harary. *Graph theory (on Demand Printing of 02787)*. CRC Press, 2018.

[284] B. Mohamed Harif and A. Nazeera Begam. On pseudo regular spherical fuzzy graphs. *Ratio Mathematica*, 50, 2023.

[285] Felix Hausdorff. *Set theory*, volume 119. American Mathematical Soc., 2021.

[286] Pavol Hell and David G Kirkpatrick. Packings by complete bipartite graphs. *SIAM Journal on Algebraic Discrete Methods*, 7(2):199–209, 1986.

[287] Pavol Hell and Jaroslav Nesetril. *Graphs and homomorphisms*, volume 28. OUP Oxford, 2004.

[288] Hendy Hendy. The h-super (anti) magic decompositions of antiprism graphs. In *AIP Conference Proceedings*, volume 1707. AIP Publishing, 2016.

[289] Petr Hliněnỳ and Gelasio Salazar. On the crossing number of almost planar graphs. In *Graph Drawing: 14th International Symposium, GD 2006, Karlsruhe, Germany, September 18-20, 2006. Revised Papers 14*, pages 162–173. Springer, 2007.

[290] Michael Hoffmann and Meghana M Reddy. The number of edges in maximal 2-planar graphs. *arXiv preprint arXiv:2303.08726*, 2023.

[291] Derek Allan Holton and John Sheehan. *The Petersen Graph*, volume 7. Cambridge University Press, 1993.

[292] Seok-Hee Hong. Algorithms for 1-planar graphs. In *Beyond Planar Graphs: Communications of NII Shonan Meetings*, pages 69–87. Springer, 2020.

[293] Seok-Hee Hong and Hiroshi Nagamochi. Testing full outer-2-planarity in linear time. In *Graph-Theoretic Concepts in Computer Science: 41st International Workshop, WG 2015, Garching, Germany, June 17-19, 2015, Revised Papers 41*, pages 406–421. Springer, 2016.

[294] W Patrick Hooper. Grid graphs and lattice surfaces. *International Mathematics Research Notices*, 2013(12):2657–2698, 2013.

[295] Alexander K Hope. A planar graph drawing program. *Software: Practice and Experience*, 1(1):83–91, 1971.

[296] Tamás Horváth, Jan Ramon, and Stefan Wrobel. Frequent subgraph mining in outerplanar graphs. In *Proceedings of the 12th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 197–206, 2006.

[297] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[298] Abrar Hussain, Ahmed Alsanad, Kifayat Ullah, Zeeshan Ali, Muhammad Kamran Jamil, and Mogeeb AA Mosleh. Investigating the short-circuit problem using the planarity index of complex q-rung orthopair fuzzy planar graphs. *Complexity*, 2021(1):8295997, 2021.

[299] R Jahir Hussain, TC Mujeeburahman, and D Dhamodharan. Some edge domination parameters in bipolar hesitancy fuzzy graph. *Ratio Mathematica*, 42:157, 2022.

[300] S Satham Hussain, R Jahir Hussain, and Ghulam Muhiuddin. *Neutrosophic vague line graphs*, volume 36. Infinite Study, 2020.

[301] S Satham Hussain, R Jahir Hussain, and Florentin Smarandache. *On neutrosophic vague graphs*. Infinite Study, 2019.

[302] Shien-Ching Hwang and Gen-Huey Chen. Two problems on butterfly graphs. In *Proceedings 1998 International Conference on Parallel and Distributed Systems (Cat. No. 98TB100250)*, pages 572–579. IEEE, 1998.

[303] Shien-Ching Hwang and Gen-Huey Chen. Cycles in butterfly graphs. *Networks: An International Journal*, 35(2):161–171, 2000.

[304] Borislav Iordanov. Hypergraphdb: a generalized graph database. In *Web-Age Information Management: WAIM 2010 International Workshops: IWGD 2010, XMLDM 2010, WCMT 2010, Jiuzhaigou Valley, China, July 15-17, 2010 Revised Selected Papers 11*, pages 25–36. Springer, 2010.

[305] Fáry István. On straight-line representation of planar graphs. *Acta scientiarum mathematicarum*, 11(229-233):2, 1948.

[306] Alon Itai, Christos H Papadimitriou, and Jayme Luiz Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, 1982.

[307] M. Mohammed Jabarulla and K. Asmiya Banu. Anti intuitionistic fuzzy graph. *Advances and Applications in Mathematical Sciences*, 20(4):597–604, 2021.

[308] Jebin Jacob, Thampy Abraham, and KP Jose. Einstein fuzzy graphs. In *AIP Conference Proceedings*, volume 2875. AIP Publishing, 2023.

[309] Deivanai Jaisankar, Sujatha Ramalingam, Nagarajan Deivanayagampillai, and Tadesse Walelign. Fuzzy outerplanar graphs and its applications. *International Journal of Computational Intelligence Systems*, 17(1):231, 2024.

[310] Bart MP Jansen, Jari JH De Kroon, and Michał Włodarczyk. Vertex deletion parameterized by elimination distance and even less. In *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1757–1769, 2021.

[311] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[312] PK Jha and G Slutzki. A note on outerplanarity of product grapi-is. *Applicationes Mathematicae*, 21(4):537–544, 1993.

[313] Jianwen Jiang, Yuxuan Wei, Yifan Feng, Jingxuan Cao, and Yue Gao. Dynamic hypergraph neural networks. In *IJCAI*, pages 2635–2641, 2019.

[314] Radu Jianu, Adrian Rusu, Andrew J Fabian, and David H Laidlaw. A coloring solution to the edge crossing problem. In *2009 13th International Conference Information Visualisation*, pages 691–696. IEEE, 2009.

[315] Paul Jungeblut. *On Interval Planar Graphs*. PhD thesis, Informatics Institute, 2017.

[316] Paul Jungeblut, Laura Merker, and Torsten Ueckerdt. A sublinear bound on the page number of upward planar graphs. *SIAM Journal on Discrete Mathematics*, 37(4):2312–2331, 2023.

[317] Michael Jünger and Petra Mutzel. *Graph drawing software*. Springer Science & Business Media, 2012.

[318] Frank Kammer. Determining the smallest k such that g is k-outerplanar. In *Algorithms–ESA 2007: 15th Annual European Symposium, Eilat, Israel, October 8-10, 2007. Proceedings 15*, pages 359–370. Springer, 2007.

[319] Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Neutrosophic graphs: a new dimension to graph theory*. Infinite Study, 2015.

[320] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Subset Vertex Multigraphs and Neutrosophic Multigraphs for Social Multi Networks*. Infinite Study, 2019.

[321] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.

[322] Mihyun Kang and Michael Missethan. The giant component and 2-core in sparse random outerplanar graphs. *arXiv preprint arXiv:2004.13319*, 2020.

[323] Haim Kaplan and Yahav Nussbaum. Maximum flow in directed planar graphs with vertex capacities. *Algorithmica*, 61(1):174–189, 2011.

[324] M. G. Karunambigai, Muhammad Akram, S. Sivasankar, and K. Palanivel. Balanced intuitionistic fuzzy graphs. *Applied mathematical sciences*, 7:2501–2514, 2013.

[325] M. G. Karunambigai, K. Palanivel, and S. Sivasankar. Edge regular intuitionistic fuzzy graph. 2015.

[326] Michael Kaufmann and Torsten Ueckerdt. The density of fan-planar graphs. *arXiv preprint arXiv:1403.6184*, 2014.

[327] M Kaviyarasu, Hossein Rashmanlou, Saeed Kosari, Said Broumi, R Venitha, M Rajeshwa, and Farshid Mofidnakhaei. Circular economy strategies to promote sustainable development using t-neutrosophic fuzzy graph. *Neutrosophic Sets and Systems*, 72:186–210, 2024.

[328] Murugan Kaviyarasu, Mohammed Alqahtani, Murugesan Rajeshwari, and Gopikumar Sudalaimuthu. Complex t-intuitionistic fuzzy graph with applications of rubber industrial water wastes. *Mathematics*, 12(13):1950, 2024.

[329] Murugan Kaviyarasu, Mohammed Alqahtani, Murugesan Rajeshwari, and Gopikumar Sudalaimuthu. Complex t-intuitionistic fuzzy graph with applications of rubber industrial water wastes. *Mathematics*, 2024.

[330] Murugan Kaviyarasu, Lumini□a-Ioana Cotîrlă, Daniel Breaz, Murugesan Rajeshwari, and Eleonora Rapeanu. A study on complex t-neutrosophic graph with intention to preserve biodiversity. *Symmetry*, 16(8):1033, 2024.

[331] Andrei V Kelarev. On undirected cayley graphs. *Australasian Journal of Combinatorics*, 25:73–78, 2002.

[332] Majid Khalili, Rajabali Borzooei, and D. Ebrahimibagha. Bipolar inverse fuzzy graphs. *Fuzzy Systems and its Applications*, 6(1):269–285, 2023.

[333] Waheed Ahmad Khan, Waqar Arif, Quoc Hung NGUYEN, Thanh Trung Le, and Hai Van Pham. Picture fuzzy directed hypergraphs with applications towards decision-making and managing hazardous chemicals. *IEEE Access*, 2024.

[334] Waheed Ahmad Khan, Khurram Faiz, and Abdelghani Taouti. Cayley picture fuzzy graphs and interconnected networks. *Intelligent Automation & Soft Computing*.

[335] Jasminara Khatun, Sk Amanathulla, and Madhumangal Pal. A comprehensive study on m-polar picture fuzzy graphs and its application. *Journal of Uncertain Systems*, page 2450016, 2024.

[336] Jasminara Khatun, Sk Amanathulla, and Madhumangal Pal. Picture fuzzy cubic graphs and their applications. *Journal of Intelligent & Fuzzy Systems*, (Preprint):1–18, 2024.

[337] Alec Kirkley, Hugo Barbosa, Marc Barthelemy, and Gourab Ghoshal. From the betweenness centrality in street networks to structural invariants in random planar graphs. *Nature communications*, 9(1):2501, 2018.

[338] A. Kirupa and J. Jesintha Rosline. Domination in double layered fuzzy graphs. *THE 11TH NATIONAL CONFERENCE ON MATHEMATICAL TECHNIQUES AND APPLICATIONS*, 2019.

[339] Irving Kittell. A group of operations on a partially colored map. *Bulletin of the American Mathematical Society*, 41(6):407–413, 1935.

[340] Victor Klee. Diameters of polyhedral graphs. *Canadian Journal of Mathematics*, 16:602–614, 1964.

[341] Philip N Klein, Shay Mozes, and Oren Weimann. Shortest paths in directed planar graphs with negative lengths: A linear-space o (n log2 n)-time algorithm. *ACM Transactions on Algorithms (TALG)*, 6(2):1–18, 2010.

[342] Philip N. Klein, Satish Rao, Monika Henzinger, and Sairam Subramanian. Faster shortest-path algorithms for planar graphs. In *Symposium on the Theory of Computing*, 1994.

[343] Krzysztof Kozminski and Edwin Kinnen. An algorithm for finding a rectangular dual of a planar graph for use in area planning for vlsi integrated circuits. In *21st Design Automation Conference Proceedings*, pages 655–656. IEEE, 1984.

[344] Evangelos Kranakis, Tim Mott, and Ladislav Stacho. Online routing in quasi-planar and quasi-polyhedral graphs. In *Fourth Annual IEEE International Conference on Pervasive Computing and Communications Workshops (PERCOMW'06)*, pages 5–pp. IEEE, 2006.

[345] Jan Kratochvíl and Aleš Kuběna. On intersection representations of co-planar graphs. *Discrete Mathematics*, 178(1-3):251–255, 1998.

[346] P. K. Kishore Kumar, S. Lavanya, Hossein Rashmanlou, and Mostafa Nouri Jouybari. Magic labeling on interval-valued intuitionistic fuzzy graphs. *J. Intell. Fuzzy Syst.*, 33:3999–4006, 2017.

[347] André Kündgen, Eric Mendelsohn, and Vitaly Voloshin. Colouring planar mixed hypergraphs. *the electronic journal of combinatorics*, 7:R60–R60, 2000.

[348] A Kuppan and J Ravi Sankar. Decomposition of fuzzy zero divisor graph. In *Journal of Physics: Conference Series*, volume 1770, page 012060. IOP Publishing, 2021.

[349] A Kuppan and J Ravi Sankar. Fuzzy zero divisor graph in a commutative ring. *TWMS Journal of Applied and Engineering Mathematics*, 11(Jaem Vol 11-Special Issue, 2021):42, 2021.

[350] Lucas Lacasa, Bartolo Luque, Fernando Ballesteros, Jordi Luque, and Juan Carlos Nuno. From time series to complex networks: The visibility graph. *Proceedings of the National Academy of Sciences*, 105(13):4972–4975, 2008.

[351] Alistair H Lachlan and Robert E Woodrow. Countable ultrahomogeneous undirected graphs. *Transactions of the American Mathematical Society*, pages 51–94, 1980.

[352] L Subha Lakshmi and NR Santhi Maheswari. A study on constant and regular hesitancy fuzzy soft graphs. *International J. Math. Combin. Vol*, 4:33–43, 2023.

[353] T. Lavanya, M. Rowthri, and D. Amsaveni. Spherical fuzzy cycle and spherical fuzzy tree. *Journal of Computational Analysis and Applications (JoCAAA)*, 33(05):610–615, 2024.

[354] Ronan Le Bras, Carla P Gomes, and Bart Selman. Double-wheel graphs are graceful. In *Twenty-Third International Joint Conference on Artificial Intelligence*, 2013.

[355] Alexandros Leivaditis, Alexandros Singh, Giannos Stamoulis, Dimitrios M Thilikos, Konstantinos Tsatsanis, and Vasiliki Velona. Minor-obstructions for apex sub-unicyclic graphs. *Discrete Applied Mathematics*, 284:538–555, 2020.

[356] Azriel Levy. *Basic set theory*. Courier Corporation, 2012.

[357] Aihua Li and Qing Wu. Interlace polynomials of ladder graphs. *Journal of Combinatorics, Information & System Sciences*, 35(1/2):243, 2010.

[358] Deng-Feng Li. Multiattribute decision making models and methods using intuitionistic fuzzy sets. *Journal of computer and System Sciences*, 70(1):73–85, 2005.

[359] Grace MX Li and Arthur LB Yang. The e-positivity of two families of $(claw, 2k\_2)$-free graphs. *arXiv preprint arXiv:1912.09717*, 2019.

[360] Hongxing Li and Vincent C Yen. *Fuzzy sets and fuzzy decision-making*. CRC press, 1995.

[361] Shasha Li, Wei Li, and Xueliang Li. The generalized connectivity of complete bipartite graphs. *arXiv preprint arXiv:1012.5710*, 2010.

[362] Yan Li and Xin Zhang. The structure and the list 3-dynamic coloring of outer-1-planar graphs. *Discrete Mathematics & Theoretical Computer Science*, 23(Graph Theory), 2021.

[363] Xiao Qun Liao, Tong Su, and Li Ma. Application of neutrosophic minimum spanning tree in electrical power distribution network. *CAAI Transactions on Intelligence Technology*, 5(2):99–105, 2020.

[364] Xiaowei Liao, Yong Xu, and Haibin Ling. Hypergraph neural networks for hypergraph matching. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pages 1266–1275, 2021.

[365] Don R. Lick and Arthur T. White. k-degenerate graphs. *Canadian Journal of Mathematics*, 22:1082 – 1096, 1970.

[366] CC Lin et al. The family of fan-planar graphs. In *WALCOM: Algorithms and Computation: 17th International Conference and Workshops, WALCOM 2023, Hsinchu, Taiwan, March 22–24, 2023, Proceedings*, volume 13973, page 12. Springer Nature, 2023.

[367] Jenq-Jong Lin and Min-Jen Jou. The number of maximal independent sets in quasi-tree graphs and quasi-forest graphs. *Open Journal of Discrete Mathematics*, 7(3):134–147, 2017.

[368] WF Lindgren. An infinite class of hypohamiltonian graphs. *The American Mathematical Monthly*, 74(9):1087–1089, 1967.

[369] Andrzej Lingas. Subgraph isomorphism for biconnected outerplanar graphs in cubic time. In *STACS 86: 3rd Annual Symposium on Theoretical Aspects of Computer Science Orsay, France, January 16–18, 1986 3*, pages 98–103. Springer, 1986.

[370] Chun-Hung Liu and Youngho Yoo. Tight minimum degree conditions for apex-outerplanar minors and subdivisions in graphs and digraphs. *arXiv preprint arXiv:2403.11470*, 2024.

[371] Pasit Lorterapong and Osama Moselhi. Project-network analysis using fuzzy sets theory. *Journal of Construction Engineering and Management*, 122(4):308–318, 1996.

[372] László Lovász and Balázs Szegedy. Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, 96(6):933–957, 2006.

[373] Anam Luqman, Muhammad Akram, and Florentin Smarandache. Complex neutrosophic hypergraphs: new social network models. *Algorithms*, 12(11):234, 2019.

[374] Rupkumar Mahapatra, Sovan Samanta, Tofigh Allahviranloo, and Madhumangal Pal. Radio fuzzy graphs and assignment of frequency in radio stations. *Computational and Applied Mathematics*, 38:1–20, 2019.

[375] Rupkumar Mahapatra, Sovan Samanta, and Madhumangal Pal. Generalized neutrosophic planar graphs and its application. *Journal of Applied Mathematics and Computing*, 65(1):693–712, 2021.

[376] NR Maheswari and C Sekar. On m-neighbourly irregular instuitionistic fuzzy graphs. *International Journal of Mathematical Combinatorics*, 3, 2016.

[377] NR Santhi Maheswari and C Sekar. On pseudo regular fuzzy graphs. *Annals of pure and Applied Mathematics*, 11(1):105–113, 2016.

[378] Pinaki Majumdar. Neutrosophic sets and its applications to decision making. In *Computational Intelligence for Big Data Analysis: Frontier Advances and Applications*, pages 97–115. Springer, 2015.

[379] Parimala Mani, Ibtesam Alshammari, and Halimah Alshehri. Shortest path algorithm of a network via spherical fuzzy digraphs. *Journal of Engineering Research*, 2021.

[380] Stephen Marshall. *Streets and patterns*. Routledge, 2004.

[381] Adolfo Paolo Masucci, Duncan Smith, Andrew Crooks, and Michael Batty. Random planar graphs and the london street network. *The European Physical Journal B*, 71:259–271, 2009.

[382] Sunil Mathew and John N Mordeson. Fuzzy influence graphs. *New Mathematics and Natural Computation*, 13(03):311–325, 2017.

[383] Colin McDiarmid, Angelika Steger, and Dominic JA Welsh. Random planar graphs. *Journal of Combinatorial Theory, Series B*, 93(2):187–205, 2005.

[384] Carver Mead and Lynn Conway. Introduction to vlsi systems, 1980.

[385] N Meenal and J Jeromi Jovita. A first look at inverse eccentric fuzzy graph. In *AIP Conference Proceedings*, volume 2829. AIP Publishing, 2023.

[386] Seema Mehra and Manjeet Singh. *Single valued neutrosophic signed graphs*. Infinite Study, 2017.

[387] Jixiang Meng. Connectivity of vertex and edge transitive graphs. *Discrete applied mathematics*, 127(3):601–613, 2003.

[388] Jixiang Meng. Optimally super-edge-connected transitive graphs. *Discrete Mathematics*, 260(1-3):239–248, 2003.

[389] Gary L Miller. Finding small simple cycle separators for 2-connected planar graphs. In *Proceedings of the sixteenth annual ACM symposium on Theory of computing*, pages 376–382, 1984.

[390] Nahid Mohajeri, Jon R French, and Michael Batty. Evolution and entropy in the organization of urban street patterns. *Annals of GIS*, 19(1):1–16, 2013.

[391] S Yahya Mohamed and R JahirHussain. Irregularity and density of irregular intuitionistic fuzzy graphs.

[392] S Yahya Mohamed and S Suganthi. Fair fuzzy matching in middle fuzzy graph. *Ratio Mathematica*, 43:176–182, 2022.

[393] B. Mohamed Harif and Nazeera Begam. On regular spherical fuzzy graph. *Journal of Hyperstructures*, 12(2):244–256, 2023.

[394] Kartick Mohanta, Arindam Dey, and Anita Pal. A study on picture dombi fuzzy graph. 2020.

[395] Bojan Mohar. Apex graphs with embeddings of face-width three. *Discrete Mathematics*, 176(1-3):203–210, 1997.

[396] Bojan Mohar. Face covers and the genus problem for apex graphs. *Journal of Combinatorial Theory, Series B*, 82(1):102–117, 2001.

[397] Bojan Mohar. Face covers and the genus problem for apex graphs. *J. Comb. Theory B*, 82:102–117, 2001.

[398] Rahul Mondal and Ganesh Ghorai. Inverse fuzzy mixed planar graphs with application. *International Journal of Applied and Computational Mathematics*, 10(4):131, 2024.

[399] Mickael Montassier, Patrice Ossona de Mendez, André Raspaud, and Xuding Zhu. Decomposing a graph into forests. *Journal of Combinatorial Theory, Series B*, 102(1):38–52, 2012.

[400] John N Mordeson and Sunil Mathew. *Advanced topics in fuzzy graph theory*, volume 375. Springer, 2019.

[401] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[402] MI Moussa and EM Badr. Ladder and subdivision of ladder graphs with pendant edges are odd graceful. *arXiv preprint arXiv:1604.02347*, 2016.

[403] G Muhiuddin, Saira Hameed, Ayman Rasheed, and Uzma Ahmad. Cubic planar graph and its application to road network. *Mathematical Problems in Engineering*, 2022(1):5251627, 2022.

[404] R Muthuraj, VV Vijesh, and S Sujith. Some peculiar dominations in intuitionistic anti-fuzzy graphs. *Malaya Journal of Matematik*, 9(01):178–184, 2021.

[405] K. K. Myithili and R. Keerthika. Types of intuitionistic fuzzy k-partite hypergraphs. 2020.

[406] K. K. Myithili and R. Keerthika. Regularity and duality of intuitionistic fuzzy k-partite hypergraphs. *Notes on Intuitionistic Fuzzy Sets*, 2023.

[407] K. K. Myithili and C. Nandhini. Mixed picture fuzzy graph. *Ratio Mathematica*, 42:225, 2022.

[408] K. K. Myithili, Rangasamy Parvathi, and Muhammad Akram. Certain types of intuitionistic fuzzy directed hypergraphs. *International Journal of Machine Learning and Cybernetics*, 7:287–295, 2016.

[409] Ayan Nandy, Sagnik Sen, and Éric Sopena. Outerplanar and planar oriented cliques. *Journal of Graph Theory*, 82, 2014.

[410] S Ravi Narayanan and S Murugesan. Isomorphic properties of edge irregular intuitionistic fuzzy graphs.

[411] Reza Naserasr and Weiqiang Yu. Separating signatures in signed planar graphs. *Discrete Applied Mathematics*, 338:302–310, 2023.

[412] Nelson Nauata and Yasutaka Furukawa. Vectorizing world buildings: Planar graph reconstruction by primitive detection and relationship inference. In *Computer Vision–ECCV 2020: 16th European Conference, Glasgow, UK, August 23–28, 2020, Proceedings, Part VIII 16*, pages 711–726. Springer, 2020.

[413] Irfan Nazeer and Tabasam Rashid. Picture fuzzy incidence graphs with application. *Punjab University Journal of Mathematics*, 2021.

[414] Irfan Nazeer and Tabasam Rashid. Connectivity concepts in intuitionistic fuzzy incidence graphs with application. *International Journal of Applied and Computational Mathematics*, 8, 2022.

[415] Jaroslav Nešetřil, Patrice Ossona de Mendez, and David R Wood. Characterisations and examples of graph classes with bounded expansion. *European Journal of Combinatorics*, 33(3):350–373, 2012.

[416] Masakazu Nihei. On the girths of regular planar graphs. *Pi Mu Epsilon Journal*, 10(3):186–190, 1995.

[417] TM Nishad, Talal Ali Al-Hawary, and B Mohamed Harif. General fuzzy graphs. *Ratio Mathematica*, 47, 2023.

[418] Takao Nishizeki and Norishige Chiba. *Planar graphs: Theory and algorithms*. Elsevier, 1988.

[419] Takao Nishizeki and Md Saidur Rahman. *Planar graph drawing*, volume 12. World Scientific, 2004.

[420] Deva Nithyanandham and Felix Augustin. A bipolar fuzzy p-competition graph based aras technique for prioritizing covid-19 vaccines. *Applied Soft Computing*, 146:110632, 2023.

[421] Bei Niu and Xin Zhang. Linear arboricity of nic-planar graphs. *Acta Mathematicae Applicatae Sinica, English Series*, 35(4):924–934, 2019.

[422] Jianbing Niu and Cun-Quan Zhang. Cliques, minors and apex graphs. *Discrete mathematics*, 309(12):4095–4107, 2009.

[423] Joakim Alme Nordstrand. Exploring graph parameters similar to tree-width and path-width. Master's thesis, The University of Bergen, 2017.

[424] Giuseppe Notarstefano and Gianfranco Parlangeli. Controllability and observability of grid graphs via reduction and symmetries. *IEEE Transactions on Automatic Control*, 58(7):1719–1731, 2013.

[425] Toto Nusantara, Desi Rahmadani, Mochammad Hafiizh, Elita Dia Cahyanti, and Abdullah Bin Gani. On vertice and edge regular anti fuzzy graphs. In *Journal of Physics: Conference Series*, volume 1783, page 012098. IOP Publishing, 2021.

[426] Anargyros Oikonomou and Antonios Symvonis. Monotone drawings of k-inner planar graphs. In *International Symposium on Graph Drawing and Network Visualization*, pages 347–353. Springer, 2018.

[427] Deryk Osthus, Hans Jürgen Prömel, and Anusch Taraz. On random planar graphs, the number of planar graphs and their triangulations. *Journal of Combinatorial Theory, Series B*, 88(1):119–134, 2003.

[428] Takashi Owaki and Takashi Machida. Roadnetgan: Generating road networks in planar graph representation. In *Neural Information Processing: 27th International Conference, ICONIP 2020, Bangkok, Thailand, November 18–22, 2020, Proceedings, Part IV 27*, pages 535–543. Springer, 2020.

[429] János Pach, Rom Pinchasi, Micha Sharir, and Géza Tóth. Topological graphs with no large grids. *Graphs and Combinatorics*, 21(3):355–364, 2005.

[430] Sakshi Dev Pandey, AS Ranadive, and Sovan Samanta. Bipolar-valued hesitant fuzzy graph and its application. *Social Network Analysis and Mining*, 12(1):14, 2022.

[431] Gianfranco Parlangeli and Giuseppe Notarstefano. On the reachability and observability of path and cycle graphs. *IEEE Transactions on Automatic Control*, 57(3):743–748, 2011.

[432] R Parvathi and MG Karunambigai. Intuitionistic fuzzy graphs. In *Computational Intelligence, Theory and Applications: International Conference 9th Fuzzy Days in Dortmund, Germany, Sept. 18–20, 2006 Proceedings*, pages 139–150. Springer, 2006.

[433] Benedict Paten, Mark Diekhans, Dent Earl, John St John, Jian Ma, Bernard Suh, and David Haussler. Cactus graphs for genome comparisons. *Journal of Computational Biology*, 18(3):469–481, 2011.

[434] T Pathinathan, J Jon Arockiaraj, and E Mike Dison. Constant hesitancy fuzzy graphs. *Global Journal of Pure and Applied Mathematics (GJPAM)*, 12(2):2016, 2016.

[435] T PATHINATHAN and A KIRUPA. Intuitionistic k-partitioned fuzzy graph.

[436] Georgios A Pavlopoulos, Maria Secrier, Charalampos N Moschopoulos, Theodoros G Soldatos, Sophia Kossida, Jan Aerts, Reinhard Schneider, and Pantelis G Bagos. Using graph theory to analyze biological networks. *BioData mining*, 4:1–27, 2011.

[437] Nicole Pearcy, Jonathan J Crofts, and Nadia Chuzhanova. Hypergraph models of metabolism. *International Journal of Biological, Veterinary, Agricultural and Food Engineering*, 8(8):752–756, 2014.

[438] Sriram Pemmaraju and Steven Skiena. *Computational discrete mathematics: Combinatorics and graph theory with mathematica®*. Cambridge university press, 2003.

[439] Iztok Peterin. A characterization of planar partial cubes. *Discrete mathematics*, 308(24):6596–6600, 2008.

[440] Maziyar Baran Pouyan and Mehrdad Nourani. Clustering single-cell expression data using random forest graphs. *IEEE journal of biomedical and health informatics*, 21(4):1172–1181, 2016.

[441] Tarasankar Pramanik, Madhumangal Pal, Sukumar Mondal, and Sovan Samanta. A study on bipolar fuzzy planar graph and its application in image shrinking. *J. Intell. Fuzzy Syst.*, 34:1863–1874, 2018.

[442] Tarasankar Pramanik, Sovan Samanta, and Madhumangal Pal. Interval-valued fuzzy planar graphs. *International journal of machine learning and cybernetics*, 7:653–664, 2016.

[443] Shio Gai Quek, Ganeshsree Selvachandran, D Ajay, P Chellamani, David Taniar, Hamido Fujita, Phet Duong, Le Hoang Son, and Nguyen Long Giang. New concepts of pentapartitioned neutrosophic graphs and applications for determining safest paths and towns in response to covid-19. *Computational and Applied Mathematics*, 41(4):151, 2022.

[444] Nahlia Rakhmawati, Agus Widodo, Noor Hidayat, and Abdul Rouf Alghofari. Optimal path with interval value of intuitionistic fuzzy number in multigraph. *IAENG International Journal of Computer Science*, 51(1):39–44, 2024.

[445] Yongsheng Rao, Ruxiang Chen, Waheed Ahmad Khan, and Alishba Zahid. Some new concepts of interval-valued picture fuzzy graphs and their application toward the selection criteria. *International Journal of Fuzzy Systems*, 2024.

[446] Hossein Rashmanlou and Madhumangal Pal. Balanced interval-valued fuzzy graphs. 2013.

[447] Hossein Rashmanlou, Sovan Samanta, Madhumangal Pal, and Rajab Ali Borzooei. Intuitionistic fuzzy graphs with categorical properties. *Fuzzy information and Engineering*, 7(3):317–334, 2015.

[448] Sreenanda Raut and Madhumangal Pal. Fuzzy intersection graph: a geometrical approach. *Journal of Ambient Intelligence and Humanized Computing*, pages 1–25, 2022.

[449] Akbar Rezaei, Tahsin Oner, Tugce Katican, Florentin Smarandache, and N Gandotra. *A short history of fuzzy, intuitionistic fuzzy, neutrosophic and plithogenic sets*. Infinite Study, 2022.

[450] MA Rifayathali, A Prasanna, and S Ismail Mohideen. Hesitancy fuzzy magic labeling graph. *The Journal of Analysis*, 27:39–53, 2019.

[451] Heike Ripphausen-Lipa, Dorothea Wagner, and Karsten Weihe. Efficient algorithms for disjoint paths in planar graphs. In *Combinatorial Optimization*, 1993.

[452] David W Roberts. Analysis of forest succession with fuzzy graph theory. *Ecological Modelling*, 45(4):261–274, 1989.

[453] Neil Robertson and Paul D. Seymour. Graph minors. i. excluding a forest. *Journal of Combinatorial Theory, Series B*, 35(1):39–61, 1983.

[454] Neil Robertson and Paul D. Seymour. Graph minors. iii. planar tree-width. *Journal of Combinatorial Theory, Series B*, 36(1):49–64, 1984.

[455] Neil Robertson and Paul D. Seymour. Graph minors. ii. algorithmic aspects of tree-width. *Journal of algorithms*, 7(3):309–322, 1986.

[456] Neil Robertson and Paul D. Seymour. Graph minors. v. excluding a planar graph. *J. Comb. Theory B*, 41:92–114, 1986.

[457] Neil Robertson and Paul D. Seymour. Graph minors. x. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991.

[458] Neil Robertson and Paul D Seymour. Graph minors. xvi. excluding a non-planar graph. *Journal of Combinatorial Theory, Series B*, 89(1):43–76, 2003.

[459] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[460] J Jesintha Rosline and T Pathinathan. Triple layered fuzzy graph. *Int. J. Fuzzy Mathematical Arhive*, 8(1):36–42, 2015.

[461] Bernat Rovira Segu. Outerplanar partial cubes. 2022.

[462] Archana S and Preethi Kuttipulackal. Analysis of various degrees and sizes in a fuzzy semigraph. *IAENG International Journal of Applied Mathematics*, 54(5):975–983, 2024.

[463] Rıdvan Şahin. An approach to neutrosophic graph theory with applications. *Soft Computing*, 23(2):569–581, 2019.

[464] Sankar Sahoo. Intuitionistic fuzzy labeling graphs. 2018.

[465] Sankar Sahoo. Colouring of mixed fuzzy graph and its application in covid19. *Journal of Multiple-Valued Logic & Soft Computing*, 35, 2020.

[466] Sankar Sahoo and Madhumangal Pal. Different types of products on intuitionistic fuzzy graphs. *Pacific Science Review A: Natural Science and Engineering*, 17(3):87–96, 2015.

[467] Sankar Sahoo and Madhumangal Pal. Intuitionistic fuzzy competition graphs. *Journal of Applied Mathematics and Computing*, 52:37–57, 2016.

[468] Sankar Sahoo and Madhumangal Pal. Intuitionistic fuzzy tolerance graphs with application. *Journal of Applied Mathematics and Computing*, 55:495–511, 2017.

[469] Sovan Samanta and Madhumangal Pal. Irregular bipolar fuzzy graphs. *arXiv preprint arXiv:1209.1682*, 2012.

[470] Sovan Samanta and Madhumangal Pal. Fuzzy planar graphs. *IEEE Transactions on Fuzzy Systems*, 23(6):1936–1942, 2015.

[471] Sovan Samanta, Madhumangal Pal, Rupkumar Mahapatra, Kousik Das, and Robin Singh Bhadoria. A study on semi-directed graphs for social media networks. *International Journal of Computational Intelligence Systems*, 14(1):1034–1041, 2021.

[472] Sovan Samanta, Madhumangal Pal, and Anita Pal. New concepts of fuzzy planar graph. *International Journal of Advanced Research in Artificial Intelligence*, 3(1):52–59, 2014.

[473] Robert Sasak. Comparing 17 graph parameters. Master's thesis, The University of Bergen, 2010.

[474] D Sasikala and B Divya. A newfangled interpretation on fermatean neutrosophic dombi fuzzy graphs. *Neutrosophic Systems With Applications*, 7:36–53, 2023.

[475] Franco Scarselli, Marco Gori, Ah Chung Tsoi, Markus Hagenbuchner, and Gabriele Monfardini. The graph neural network model. *IEEE transactions on neural networks*, 20(1):61–80, 2008.

[476] Edward R. Scheinerman. *Intersection Classes and Multiple Intersection Parameters of a Graph*. PhD thesis, Princeton University, 1984. As cited by Brandstädt, Le & Spinrad (1999).

[477] Thomas Schneck. *New parameters for beyond-planar graphs*. PhD thesis, Universität Tübingen, 2020.

[478] Walter Schnyder. Embedding planar graphs on the grid. In *Proceedings of the first annual ACM-SIAM symposium on Discrete algorithms*, pages 138–148, 1990.

[479] Carsten R Seemann, Vincent Moulton, Peter F Stadler, and Marc Hellmuth. Planar median graphs and cubesquare-graphs. *Discrete Applied Mathematics*, 331:38–58, 2023.

[480] Mushtaq Ahmad Shah. A study on hypo hamiltonian graphs. *African journal of Mathematics and computer Science research*, 6(2):16–19, 2013.

[481] Khushro Shahookar and Pinaki Mazumder. Vlsi cell placement techniques. *ACM Computing Surveys (CSUR)*, 23(2):143–220, 1991.

[482] Gulfam Shahzadi, Muhammad Akram, and Bijan Davvaz. Pythagorean fuzzy soft graphs with applications. *Journal of Intelligent & Fuzzy Systems*, 38(4):4977–4991, 2020.

[483] Anthony Shannon and Krassimir Atanassov. On intuitionistic fuzzy multigraphs and their index matrix interpretations. In *2004 2nd International IEEE Conference on'Intelligent Systems'. Proceedings (IEEE Cat. No. 04EX791)*, volume 2, pages 440–443. IEEE, 2004.

[484] Ayesha Shareef, Uzma Ahmad, Saba Siddique, and Mohammed M. Ali Al-Shamiri. Pythagorean fuzzy incidence graphs with application in illegal wildlife trade. *AIMS Mathematics*, 2023.

[485] Yossi Shiloach and Shimon Even. An on-line edge-deletion problem. *Journal of the ACM (JACM)*, 28(1):1–4, 1981.

[486] Prem Kumar Singh. Interval-valued neutrosophic graph representation of concept lattice and its (α, β, γ)-decomposition. *Arabian Journal for Science and Engineering*, 43(2):723–740, 2018.

[487] Prem Kumar Singh. Turiyam set a fourth dimension data representation. *Journal of Applied Mathematics and Physics*, 9(7):1821–1828, 2021.

[488] Prem Kumar Singh. Complex plithogenic set. *International Journal of Neutrosophic Sciences*, 18(1):57–72, 2022.

[489] Prem Kumar Singh et al. Dark data analysis using intuitionistic plithogenic graphs. *International Journal of Neutrosophic Sciences*, 16(2):80–100, 2021.

[490] S Sivasankar and Said Broumi. Balanced neutrosophic graphs. *Neutrosophic Sets and Systems*, 50:309–319, 2022.

[491] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[492] Florentin Smarandache. n-superhypergraph and plithogenic n-superhypergraph. *Nidus Idearum*, 7:107–113, 2019.

[493] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[494] Florentin Smarandache. Decision making based on valued fuzzy superhypergraphs. 2023.

[495] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[496] Florentin Smarandache and Mohamed Abdel-Basset. *Optimization Theory Based on Neutrosophic and Plithogenic Sets*. Academic Press, 2020.

[497] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[498] Florentin Smarandache and Nivetha Martin. *Plithogenic n-super hypergraph in novel multi-attribute decision making*. Infinite Study, 2020.

[499] Suber Bathusha SN and S Angelin Kavitha Raj. Climatic analysis based on interval-valued complex pythagorean fuzzy graph structure. *Annals of Mathematics and Computer Science*, 19:10–33, 2023.

[500] Alexander Soifer. *The mathematical coloring book: Mathematics of coloring and the colorful life of its creators*. Springer, 2009.

[501] Metawee Songsaeng and Aiyared Iampan. *Neutrosophic set theory applied to UP-algebras*. Infinite Study, 2019.

[502] VS Sreedevi and Bloomy Joseph. Degree of intuitionistic l-fuzzy graph. *J. Math. Comput. Sci.*, 11(1):1011–1017, 2021.

[503] Ben C Stöver and Kai F Müller. Treegraph 2: combining and visualizing evidence from different phylogenetic analyses. *BMC bioinformatics*, 11:1–9, 2010.

[504] M Subbulakshmi and I Valliammal. Decomposition of wheel graphs into stars, cycles and paths. *Malaya Journal of Matematik*, 9(01):456–460, 2021.

[505] A Sudha and P Sundararajan. Robust fuzzy graph. *Ratio Mathematica*, 46, 2023.

[506] S Sudha, Nivetha Martin, and Florentin Smarandache. *Applications of Extended Plithogenic Sets in Plithogenic Sociogram*. Infinite Study, 2023.

[507] S Suganthi and S Yahya Mohamed. Fair fuzzy matching in middle fuzzy graph.

[508] A Sugumaran and K Rajesh. Sum divisor cordial labeling of herschel graph. *Annals of pure and applied mathematics*, 14(3):465–472, 2017.

[509] Andrew Suk. k-quasi-planar graphs. In *International Symposium on Graph Drawing*, pages 266–277. Springer, 2011.

[510] Andrew Suk and Bartosz Walczak. New bounds on the maximum number of edges in k-quasi-planar graphs. *Computational Geometry*, 50:24–33, 2015.

[511] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[512] R Sumathi, R Sujatha, and R Sundareswaran. Best fuzzy graph with application in fuzzy social network. In *AIP Conference Proceedings*, volume 2393. AIP Publishing, 2022.

[513] Weigang Sun, Shuai Wang, and Jingyuan Zhang. Counting spanning trees in prism and anti-prism graphs. *J. Appl. Anal. Comput*, 6(1):65–75, 2016.

[514] Tintumol Sunny, Pramada Ramachandran, and Sr Magie Jose. On level subsets of intuitionistic l-fuzzy graphs. *Malaya Journal of Matematik*, 8(04):1942–1946, 2020.

[515] Yusuke Suzuki. 1-planar graphs. In *Beyond Planar Graphs: Communications of NII Shonan Meetings*, pages 47–68. Springer, 2020.

[516] Maciej M. Sysło and Andrzej Proskurowski. On halin graphs. In *Graph Theory: Proceedings of a Conference held in Lagów, Poland, February 10–13, 1981*, volume 1018 of *Lecture Notes in Mathematics*, pages 248–256. Springer-Verlag, 1983.

[517] Eulalia Szmidt. *Distances and similarities in intuitionistic fuzzy sets*, volume 307. Springer, 2014.

[518] AL-Hawary Talal and Bayan Hourani. On intuitionistic product fuzzy graphs. *Italian Journal of Pure and Applied Mathematics*, page 113.

[519] AA Talebi. Cayley fuzzy graphs on the fuzzy groups. *Computational and Applied Mathematics*, 37:4611–4632, 2018.

[520] Roberto Tamassia. *Handbook of graph drawing and visualization*. CRC press, 2013.

[521] Evelyn L Tan. Some notes on cycle graphs. *Discrete mathematics*, 105(1-3):221–226, 1992.

[522] Abolfazl Tehranian and Hamid Reza Maimani. A study of the total graph. 2011.

[523] Dušan Teodorović. Fuzzy sets theory applications in traffic and transportation. *European Journal of Operational Research*, 74(3):379–390, 1994.

[524] Carsten Thomassen. Every planar graph is 5-choosable. *J. Comb. Theory B*, 62:180–181, 1994.

[525] Jing-jing TIAN. Existence of light kite in nic-planar graphs. *Journal of Southwest China Normal University (Natural Science Edition)*, 45(4):13–20, 2020.

[526] Jingjing Tian and Xin Zhang. Pseudo-outerplanar graphs and chromatic conjectures. *Ars Comb.*, 114:353–361, 2014.

[527] Duc Long Tran. *Expanding the Graph Parameter Hierarchy*. PhD thesis, Institute of Software, 2022.

[528] Nenad Trinajstic. *Chemical graph theory*. CRC press, 2018.

[529] S Tsukiyama, T Maruyama, S Shinoda, and I Shirakawa. A condition for a maximal planar graph to have a unique rectangular dual and its application to vlsi floor-plan. In *1989 IEEE International Symposium on Circuits and Systems (ISCAS)*, pages 931–934. IEEE, 1989.

[530] Shuji Tsukiyama, Keiichi Koike, and Isao Shirakawa. An algorithm to eliminate all complex triangles in a maximal planar graph for use in vlsi floorplan. In *Algorithmic Aspects Of VLSI Layout*, pages 321–335. World Scientific, 1993.

[531] William Thomas Tutte. A theory of 3-connected graphs. *Indag. Math*, 23(441-455):8, 1961.

[532] Vakkas Uluçay. Q-neutrosophic soft graphs in operations management and communication network. *Soft Computing*, 25(13):8441–8459, 2021.

[533] Vakkas Uluçay and Memet Şahin. Intuitionistic fuzzy soft expert graphs with application. *Uncertainty discourse and applications*, 1(1):1–10, 2024.

[534] HD Nataraj Urs, RC Biradar, SS Manvi, KM Sudharshan, and KM Rudrappa. Floor plan analysis in vlsi design process: Multi-agent based approach. In *2010 International Conference on Communication and Computational Intelligence (INCOCCI)*, pages 279–283. IEEE, 2010.

[535] Masoud Valafar, Reza Rejaie, and Walter Willinger. Beyond friendship graphs: a study of user interactions in flickr. In *Proceedings of the 2nd ACM workshop on Online social networks*, pages 25–30, 2009.

[536] Seema Varghese, Severino V Gervacio, and Ambat Vijayakumar. On cycle graphs. *Asian-European Journal of Mathematics*, 16(03):2350036, 2023.

[537] Athanasios Voulodimos, Nikolaos Doulamis, Anastasios Doulamis, and Eftychios Protopapadakis. Deep learning for computer vision: A brief review. *Computational intelligence and neuroscience*, 2018(1):7068349, 2018.

[538] Donald K Wagner. Delta–wye reduction of almost-planar graphs. *Discrete Applied Mathematics*, 180:158–167, 2015.

[539] Hafidhyah Dwi Wahyuna and Diari Indriati. On the total edge irregularity strength of generalized butterfly graph. *Journal of Physics: Conference Series*, 1008, 2018.

[540] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[541] Carolyn Watters and Michael A Shepherd. A transient hypergraph-based model for data access. *ACM Transactions on Information Systems (TOIS)*, 8(2):77–102, 1990.

[542] Eric W. Weisstein. Tutte's graph. From MathWorld–A Wolfram Web Resource.

[543] Douglas Brent West et al. *Introduction to graph theory*, volume 2. Prentice hall Upper Saddle River, 2001.

[544] Douglas R White and Karl P Reitz. Graph and semigroup homomorphisms on networks of relations. *Social Networks*, 5(2):193–234, 1983.

[545] John C Wierman and Robert M Ziff. Self-dual planar hypergraphs and exact bond percolation thresholds. *arXiv preprint arXiv:0903.3135*, 2009.

[546] R Wilson. Poussin graph. 2022.

[547] Robin J Wilson. *Introduction to graph theory*. Pearson Education India, 1979.

[548] Wayne Wolf. *Modern VLSI design: system-on-chip design*. Pearson Education, 2002.

[549] Elliot S Wolk. The comparability graph of a tree. *Proceedings of the American Mathematical Society*, 13(5):789–795, 1962.

[550] Zonghan Wu, Shirui Pan, Fengwen Chen, Guodong Long, Chengqi Zhang, and S Yu Philip. A comprehensive survey on graph neural networks. *IEEE transactions on neural networks and learning systems*, 32(1):4–24, 2020.

[551] Mingji Xia, Peng Zhang, and Wenbo Zhao. Computational complexity of counting problems on 3-regular planar graphs. *Theoretical Computer Science*, 384(1):111–125, 2007.

[552] Keyulu Xu, Weihua Hu, Jure Leskovec, and Stefanie Jegelka. How powerful are graph neural networks? *arXiv preprint arXiv:1810.00826*, 2018.

[553] Zeshui Xu. *Hesitant fuzzy sets theory*, volume 314. Springer, 2014.

[554] Lanzhen Yang and Hua Mao. Intuitionistic fuzzy threshold graphs. *J. Intell. Fuzzy Syst.*, 36:6641–6651, 2019.

[555] Yiding Yang, Xinchao Wang, Mingli Song, Junsong Yuan, and Dacheng Tao. Spagan: Shortest path graph attention network. *arXiv preprint arXiv:2101.03464*, 2021.

[556] Mihalis Yannakakis. Node-and edge-deletion np-complete problems. In *Proceedings of the tenth annual ACM symposium on Theory of computing*, pages 253–264, 1978.

[557] Mihalis Yannakakis. Edge-deletion problems. *SIAM Journal on Computing*, 10(2):297–309, 1981.

[558] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[559] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[560] Ping Zhang and Gary Chartrand. Introduction to graph theory. *Tata McGraw-Hill*, 2:2–1, 2006.

[561] Xin Zhang and Bi Li. Linear arboricity of outer-1-planar graphs. *Journal of the Operations Research Society of China*, 9:181–193, 2021.

[562] Wufan Zhao, Claudio Persello, and Alfred Stein. Extracting planar roof structures from very high resolution images using graph neural networks. *ISPRS Journal of Photogrammetry and Remote Sensing*, 187:34–45, 2022.

[563] Jie Zhou, Ganqu Cui, Shengding Hu, Zhengyan Zhang, Cheng Yang, Zhiyuan Liu, Lifeng Wang, Changcheng Li, and Maosong Sun. Graph neural networks: A review of methods and applications. *AI open*, 1:57–81, 2020.